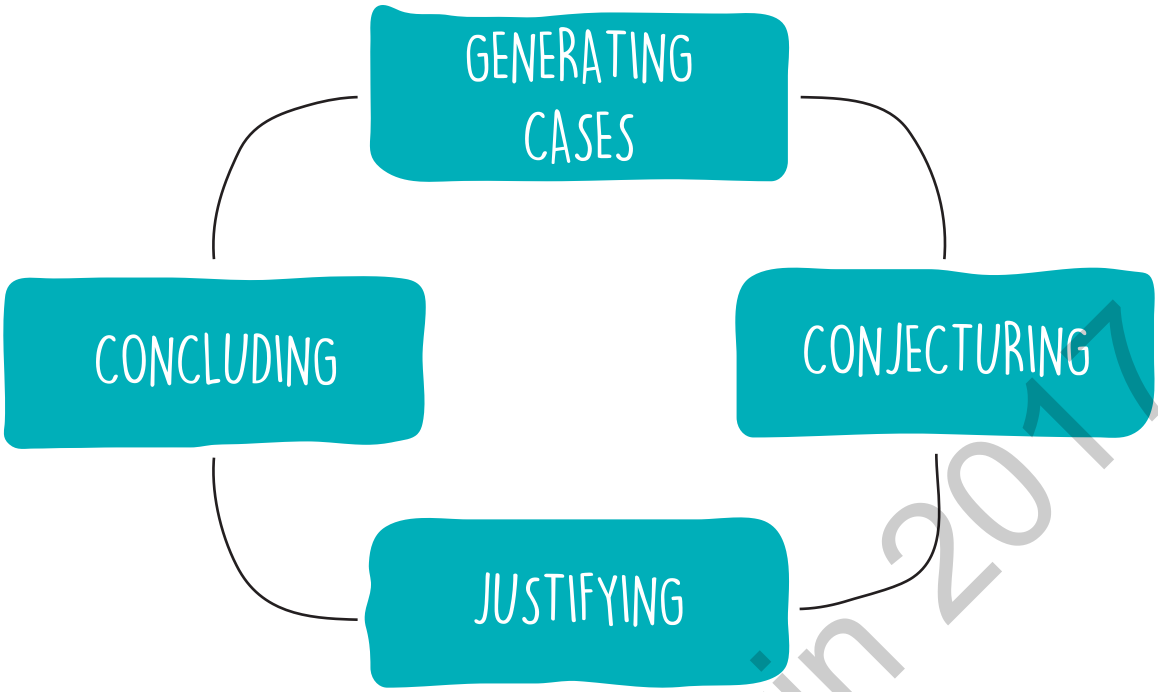


CHAPTER 1

Mathematical Argumentation

Why and What

In this chapter, you will learn

- What argumentation is and what it is not
- How to use a four-part model of argumentation: generating cases, conjecturing, justifying, and concluding
- About argumentation as a social process
- Why teaching is disciplined improvisation and how improvisation supports argumentation, norm setting, and student engagement
- Steps for introducing argumentation in your mathematics classroom
- About argumentation *in* lessons and argumentation lessons
- How to share new ideas for teaching mathematical argumentation in working together with your colleagues

ARGUMENTATION IS IMPORTANT!

> There are things that we need to communicate in everyday life, especially in the society in which we find ourselves now with all kinds of complexities. If we could just step back and think critically about it, then we should be able to come up with some kind of solution to the problems we face. That's why I just think that math argumentation is so great, not only for education, but so that you will be able to function as a human being and a citizen in this society.
>
> —Seventh-grade mathematics teacher

That's what a middle school mathematics teacher, one of our workshop participants, had to say about the potential for mathematical argumentation to make an impact outside of the classroom. What if students can use the same kind of reasoning to solve problems in their lives as they do to, say, establish that the sum of two odd numbers is an even number? As we consider our students' futures, the kind of careful reasoning that they do together in classroom mathematical argumentation is an important 21st century workplace and life skill. Making logical connections among abstract ideas and interacting with others to clarify their ideas are both deemed necessary in an increasing number of good jobs (Partnership for 21st Century Skills, 2008).

And then there are current mathematics standards. You probably know about the emphasis on mathematical practices or processes in most current state standards, including the practices that students "construct viable arguments and critique the reasoning of others" (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) and "justify mathematical ideas and arguments using precise mathematical language in written or oral communication" (Texas Education Agency, 2012). The ability to make sense of the story mathematics tells, construct a viable argument about that story, justify one's reasoning, and critique the reasoning of others are essential skills in almost every line of work and in citizen participation.

In addition to these practical considerations, we believe that the practice of mathematical argumentation is the most important of the mathematical practices because it is the fundamental way in which mathematicians communicate with each other. The search for mathematical truth is ongoing, as mathematicians create new ideas and justify them and as students reason together in a classroom.

Furthermore, access to mathematical argumentation is an equity issue. Every student should have access to this high-level disciplinary practice. Providing this access in elementary and middle school puts students on a path to higher level mathematics in high school and college. Current research indicates that about a third of the difference in mathematics achievement between students of color and white students, and between students from low- and high-income families, is attributable to the *opportunity to learn high-level mathematics* that they are given in class (Schmidt, Burroughs, Zoido, & Houang, 2015). The techniques in this book provide practices that support equitable access to high-level mathematics.

Although argumentation is serious business, it's also true that engaging in argumentation can make your mathematics classroom more joyful. Students get to play with mathematical ideas and take ownership of them in a way that often delights them. You'll most likely feel a boost yourself. One participant in our early workshops proclaimed that every Friday was argumentation day, and her class eagerly looked forward to it. While we advocate including argumentation most days, not just Fridays, we appreciated the spirit of her designation and found it a positive step in her own professional development.

The approach, techniques, and activities in this book were developed while working with teachers in a variety of settings. We have worked, in particular, with teachers in urban schools with high proportions of youth of color and students from low-income families. We have also worked in schools in more affluent communities. Teachers in all of these settings have used these methods to bring argumentation to their classrooms. Additionally, we have worked with teachers of students who receive special education services and students who are English Language Learners, and they have found that their students, too, can participate in mathematical argumentation. The vignettes and examples we present are informed by what we learned from these teachers but do not represent any one teacher.

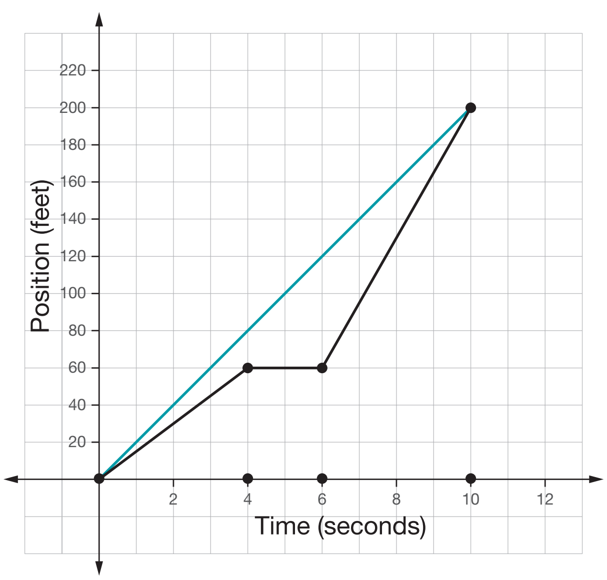
WHAT ARGUMENTATION IS—AND IS NOT

To understand what mathematical argumentation is, it is important to understand what it is not. We can contrast it with other mathematical practices and processes. Take, for example, a graph of distance as a function of time, as shown in Figure 1.1. It can be the starting place for lessons on problem solving, modeling, or argumentation, depending on the prompt that goes with the graph. A problem-solving prompt, for example, is "Create a trip with three segments that ends at 200 feet." It calls for a solution, carefully reasoned but not necessarily an argument. On the other hand, a prompt that calls for argumentation goes like this: "Raj says that if one line is steeper than another, then it represents a faster motion. Is this always true?" Notice the question, "Is this always true?" In this book, we help you develop a repertoire of ways to use that simple question, among others, to engage your students in building arguments throughout the school year.

A second question asks for argumentation: "How do we know it is true?" In this question, the focus is on a public demonstration of why a statement is true or false. The onus is on students to come up with an argument that is convincing to others. This press for truth is key in fostering argumentation that takes place *among* students so that students not only construct arguments but also critique each other's reasoning.

This approach to argumentation positions it as a social practice—what we engage in to find out the truth together (Thurston, 1998). For example, you can tell students that the area of a parallelogram is calculated by multiplying the lengths of the base and height. What if students multiply the base, height,

FIGURE 1.1 Time Versus Position Graph



and the other side length to find the area? You could simply tell them this is wrong. But it is more powerful for students to explain to each other why multiplying these three numbers together does not make sense, calling on the concept of area as a measure of two-dimensional space.

Students will likely also need help understanding what argumentation is and what it is not. They may bring their own notions of what an argument is—for example, a fight—and it will take some work to help them develop a new way of thinking about argumentation as a mathematical practice, as a way of reasoning together about the truth. We'll have more to say about classroom norms for argumentation in the following chapters, but this may be the most fundamental norm of all: *We are finding out the truth together.*

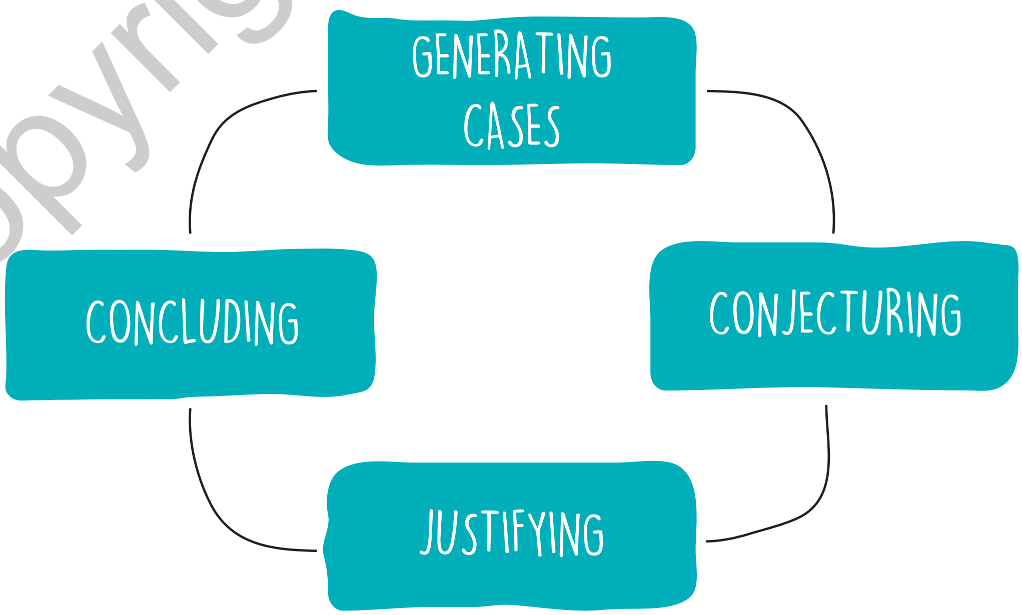
A FOUR-PART MODEL OF ARGUMENTATION

Our model of argumentation is based on what mathematicians do and what philosophers and educators have posited as parts of argumentation. We distilled the experts' (e.g., Harel & Sowder, 1998, 2007; Krummheuer, 1995; Lakatos, 1976) views on argumentation into a structure that works for teachers and students just beginning with argumentation as well as for those with more experience. The model has four parts:

- Generating cases—creating something to argue about
- Conjecturing—making bold claims
- Justifying—building a chain of reasoning
- Concluding—closure on truth or falsity

In practice, the parts of the model may get mixed together, and the process is often iterative. When you are starting out, it's useful to think of each part as a separate activity (Figure 1.2).

FIGURE 1.2 Four-Part Model for Argumentation



Important to our model is that you can make some teaching moves to elicit students' mathematical work in each part. Already, we've introduced two teaching moves—asking "Is this always true?" and "How do we know it is true?"—that you can use to stimulate mathematical argumentation among your students. Obviously, there's more to it than that. Next, we'll walk through the model for argumentation that you can use so that students get to experience all the important parts of an argument.

A *move* is the smallest piece of behavior that can be aimed at a purpose. We discuss teaching moves throughout this book as questions you can ask or actions you can take, along with their specific purposes.

Generating Cases—Creating Something to Argue About

In order for students to make a mathematical argument, they need some mathematics to argue about. That may seem pretty obvious, but the challenge is in finding the right tasks. If you have focused on problem-solving lessons or supporting rich discussion in your class, you've probably used activities in which students do what we call "generating cases." The cases often reveal patterns that students are asked to describe and explain. For example, when considering the sums of even and odd numbers—a way to introduce variables—students can explore the results of adding different numbers such as 5 + 8 as an example of odd plus even and 6 + 10 as an example of even plus even. As students come up with examples to try, they are, in the terms of our model, generating cases. In a set of cases, they can observe patterns. Often, lessons that include patterning stop with recognizing the pattern. But argumentation is just beginning as students recognize patterns across cases. Students' descriptions of patterns can be clearly articulated as conjectures, which is the next part of argumentation.

We say more about what makes for good generation of cases in Chapter 2, as well as provide tasks well suited to this purpose.

Conjecturing—Making Bold Claims

Conjectures are mathematical statements that can be determined to be true or false. They often have a level of generality that goes beyond a single case. An important class of general conjectures includes the answers to the question, "What must *always* be true?" But specific conjectures can be important as well. Specific conjectures might be about the solution to a particular equation or an observation about a regular polygon. When specific or general conjectures are appropriate is discussed more in Chapter 3.

Conjectures are mathematical statements that can be determined to be true or false.

> One easy way I can get students to make different conjectures, even in a lesson on procedures, is by asking for consensus on answers to questions, before I give the answer myself.
>
> —Sixth-grade teacher

Keep in mind that conjecturing is a time for students to make bold claims that go beyond the obvious. You don't want the only arguments that your class makes to be about statements that students already know are true; it takes the adventure out of argumentation, as well as some of the purpose. Also, arguing for a conjecture that turns out to be false can be a powerful learning activity, lending more insight into the concept behind the conjecture than might otherwise be accessible to your students. When students are conjecturing, you want to encourage bold claims with questions such as "What do you think *might* be true about all the cases you have looked at so far?"

In Chapter 3, we explain ways to elicit bold conjectures, ways to handle multiple conjectures, and ways to start a lesson with a single conjecture you select in advance with a particular purpose in mind.

Justifying—Building a Chain of Reasoning

Justifying occurs when students present reasons for why a conjecture is true or false.

A justification is a logically connected chain of statements that begins with what students already know to be true and ends by establishing the truth or falsity of a conjecture. When students justify a conjecture, they are presenting the reasons why it is true, and those reasons need to form a connection between what is known and what is yet to be known. Don't expect, though, that a justification comes out in perfect order in the social process of classroom argumentation. The logical chain may not be obvious and will have to be articulated in the next phase, concluding.

The simplest justification is a counterexample—a single example that shows that a conjecture is false. For example, after establishing that the sum of two even numbers is even, students may conjecture that

> The sum of two odd numbers must always be odd.

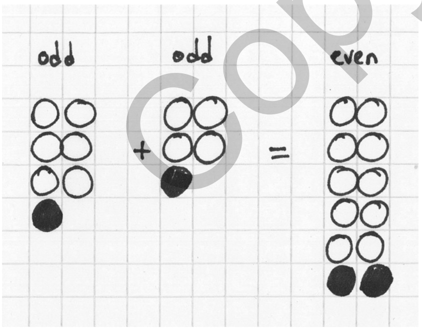
Say a student points out that $3 + 5 = 8$. That one example tells us that the sum is *not*, in fact, always odd, thereby serving as a counterexample, which is a complete justification that the conjecture is false.

A counterexample often suggests the next conjecture: In this case,

> The sum of two odd numbers is always even.

There are several ways to make a mathematically sound justification for this conjecture using different representations. The choice of representations is important in justifying; students need to be somewhat comfortable with the representation they are using, although they don't have to be expert with it. Students can make good justifications using very simple representations. For example, one justification that the sum of two odd numbers is even could rely on diagrams (Figure 1.3).

FIGURE 1.3 Justification Using a Pictorial Representation



(For more on pictorial representations in justification, see Schifter [2009] and our Chapter 5.)

> Each odd number can be represented as a set of paired dots plus one extra dot. When two odd numbers are added together, the result shows all the pairs of dots and the two extra dots that form another pair. Because the number can be represented with pairs of dots only, it is even.

Other students may use variables in their justification. They can represent the odd numbers as

> $2n + 1$ and $2m + 1$, where n and m could be any whole numbers.

Then students can argue that any time you add two odd numbers together, you will get the sum of

$$2n + 1 + 2m + 1 = 2n + 2m + 1 + 1 = 2n + 2m + 2 = 2(n + m + 1).$$

> This means that the sum of two odd numbers is a multiple of 2, so it must be even.

The statements start with what is known—a way to represent odd and even numbers—and the justification advances, in a couple of steps, to the truth of the conjecture.

These two example justifications point to how the process can lead to learning new concepts and skills. Say you are teaching sixth graders who are just learning to use algebraic expressions. Those who haven't fully assimilated this process will likely be more comfortable with something like dot diagrams for use in their justifications. If other students make the argument with variables, then you can elicit connections between the two that help all students better understand the representations. If students cannot make the symbolic argument themselves, then the lesson becomes an opportunity to introduce the use of algebraic expressions in this context, as well as an opportunity to introduce factoring of algebraic expressions. This is an example of how argumentation is not an add-on but a powerful way to engage students in content learning. We know that this is important to teachers, who have a great deal of content to cover between September and June. Argumentation shouldn't be one more thing to teach but instead should be part of a coherent practice that enables students to learn content and practices at the same time.

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Chapter 4 further explains using variations on "How do we know it is true?" to begin justifications and provides ways to keep justifications going until they are complete.

Concluding—Closure on Truth or Falsity

The conclusion of an argument is agreement on the truth or falsity of a conjecture based on a justification. Concluding can be done by an individual, but in classroom argumentation, we are looking for the whole class to come to agreement based on the justifications that the students have made together or that are based on one student's argument that has been thoroughly critiqued by the other students. The process of concluding has two parts: (1) deciding whether a conjecture is true or false, based on its justification, and (2) summarizing the justification of the conjecture in logical order.

Concluding has two parts: (1) deciding whether a conjecture is true or false and (2) summarizing the justification in logical order.

Leading a class in concluding can be challenging, so we provide several methods to accomplish it in Chapter 7.

ABOUT TRUTH

As teachers, it's worth thinking about what we mean by "true" in mathematics. There are at least two perspectives that can be taken. First, as we have done in this book, we can think of mathematical argumentation as a social process; we say that the truth is established by the agreement of a mathematics community—your class full of students. In this social-process view, mathematical truth depends on what people do together. On the other hand, mathematical statements are often thought of as true regardless of whether they have been accepted as such by any particular community. In this more absolutist view, that the sum of the measures of the interior angles of a triangle is 180° is not up for debate. But it is up for justification and shouldn't, in an argumentation-oriented mathematics class, be considered true until it is justified. For one thing, it depends on how we agree to measure angles and define interior angles. Many of us tend to hold both views simultaneously—mathematics as enduring truth and mutually agreed-upon truth. You may not need to explain these points explicitly to students, but it's good to be aware of them.

What a class decides is true depends on what students already know. For example, a second-grade class acting as a mathematical community may conclude that "the sum of two numbers is always larger than either number" because students likely only know about whole numbers, for which the statement is true. But in middle school, they can revisit that statement once signed numbers have been introduced and refine their conclusion to refer to positive numbers only. At times, you may step in, representing the larger mathematical community outside the classroom. For example, students will probably need exposure to an introductory argument about the existence and meaning of irrational numbers before they can argue about such numbers for themselves.

TEACHING AS DISCIPLINED IMPROVISATION

It's clear that teaching for argumentation calls for thinking on your feet. Even with the example we've looked at so far, students' arguments for why the sum of two odd numbers must be even could come in quite a range, from pictorial to symbolic. You can't know for sure which will come up, and students may surprise you with an approach you hadn't considered. For this reason, we say that teaching for argumentation is improvisational. You may have seen improv performers on stage or television. The performers must work together to create a scene that has never been done before but that makes sense given audience cues. Teachers are certainly not mere performers, but they are called on to create something new each day—a lesson that must make sense and lead to learning while taking into account unforeseen student ideas that will be different from student to student, class to class, and topic to topic.

Disciplined improvisation relies on knowledge of mathematics, argumentation, and teaching moves to support argumentation.

Others have claimed, and we agree, that teaching calls for *disciplined improvisation* (Sawyer, 2004). We believe this term honors the complex work of teaching in which you engage. We've described what makes teaching improvisational. But improvisation does not mean anything goes—good improvisation in any field, from theater to music to teaching, builds on strong foundational knowledge and practices. Disciplined improvisation for teaching relies on several types of knowledge that are important: knowledge of mathematics, as seen through the perspective of teaching; knowledge of the structure and process of argumentation; and knowledge of teaching moves to support argumentation. This book addresses all these types of knowledge and will guide your own disciplined improvisational teaching of argumentation, yielding some predictability to the enterprise while helping you stay open to unforeseen student contributions.

Throughout this book, we provide opportunities for you to develop your disciplined, improvisational teaching by acquiring new teaching moves and building your own mathematics knowledge that is directly relevant to your teaching. Specific moves and categories of moves are presented in separate chapters for each part of argumentation. There are opportunities for you to try out the parts of argumentation for yourself with colleagues, as well as discussions of the mathematics relevant to the multiple classroom tasks you will find in each chapter.

IMPROVISATION FOR ARGUMENTATION AND NORM SETTING

Not only is teaching improvisational, but so, too, is mathematical argumentation. Cook (2015) describes the discovery of the famous mathematician and Fields medal winner Terence Tao:

> The ancient art of mathematics, Tao has discovered, does not reward speed so much as patience, cunning and, perhaps most surprising of all, the sort of gift for collaboration and improvisation that characterizes the best jazz musicians.

Both professional mathematicians and students must be prepared to build on the ideas of others in unforeseen ways as they uncover the mathematical truth together. As you well know, students may not come to your class with this view of mathematics. Supporting argumentation in your classroom requires developing new norms. A core set of norms needs to be in place for any argumentation to happen:

- ★ Treat each other with respect.
- ★ Speak so that everyone can hear.
- ★ It's OK to make mistakes; in fact, mistakes lead to learning.
- ★ Give your own ideas and build off the ideas of others.
- ★ Discuss ideas, not people.
- ★ We are finding out the mathematical truth together.

These norms are derived from work on accountable talk (Chapin, O'Connor, & Anderson, 2003). All the norms we suggest are indicated with stars (★) throughout the book.

> Whether it is a small group or whole class, I will remind disruptive students, "Let's check back on the norms: One person speaks at a time; be respectful."
>
> —Seventh-grade teacher

Brief *warm-up games*—even nonmathematical ones—are a fun, low-risk way for all students to become familiar with the norms of mathematical argumentation.

To help students develop a new view of mathematics as well as learn to engage in argumentation improvisationally, we provide specific norms for each stage of argumentation (Figure 1.4 shows a poster available for download at resources .corwin.com/mathargumentation) and promote the use of warm-up games that are based on the improv games that improv actors use to learn their craft. These games are widely used, and we have chosen and adapted them to address norms. There are many books, as listed in our references, and websites that you can find that describe improv games. Warm-up games have simple rules of interaction that foster spontaneity and strong collaboration among participants. They also encourage playfulness, which leads to the joy in argumentation. The games can be used at the beginning of selected class periods to introduce productive norms for that lesson. We have found that introducing the norms in the sometimes nonmathematical context of games can be important for students who may have never before participated in group discussions about the truth of mathematics. The games bridge between students' everyday experiences and an academic practice in a lively, engaging way that is safe even for those with little confidence in their mathematical competence. Each game should take 5 to 10 minutes at the beginning of a class period and will ultimately save you time if students experience a norm through a game. Teachers have found that a discussion to make explicit the connection between norm and game is essential. Playing the game grounds that discussion in students' own experiences, making it easier to establish a new way of behaving in mathematics class.

FIGURE 1.4 Argumentation Norms Poster

HOW TO DO MATH ARGUMENTATION

Generating Cases

- Think about more than just one case.
- Be creative:
  - Try simple numbers or shapes.
  - Try hard numbers or shapes.
  - Try "weird" numbers or shapes.

Conjecturing

- Use patterns to make statements about what will always be true.
- Make bold conjectures about what might be true.
- Avoid judging other people's conjectures.

Justifying

- Look for reasons why a conjecture is true or false.
- Build off of other people's ideas.
- Try to convince others of your ideas, but keep in mind that you could be wrong—which is OK.
- Show it a different way. Make a drawing, table, or graph.
- Be obvious.

Concluding

- Know when to stop.
- Retell the argument from beginning to end.
- Base your conclusions on what is said, not who said it.

online resources Available for download at **resources.corwin.com/mathargumentation**

Zip, Zap, Zop is a classic simple warm-up game (e.g., Lobman & Lundquist, 2007) helps students understand that it is OK to make mistakes, speak so that everyone can hear, and pay close attention to one another.

ZIP, ZAP, ZOP

- Players stand in a circle so that everyone can see everyone else.
- The players throw an imaginary ball to one another within the circle, saying "zip," "zap," or "zop" (one with each throw, in that order, repeating the sequence until the game is over).
- The first player starts by throwing a "zip" to someone else in the circle.
- The catcher then becomes the thrower and throws the "zap" to someone else in the circle.
- Players continue, in any order, until most have had a turn.

Inevitably, someone will say the wrong word, and that can prompt what is known as the "circus bow." In the circus, when acrobats make a mistake and land in the net, they do not slink off in embarrassment. Instead, they jump off the net and make a bow with a flourish, as if they had intended the fall all along. Students can take a circus bow whenever they "mess up." This helps reinforce dramatically that making mistakes is OK.

Players in Zip, Zap, Zop must pay attention to each other to see where the "ball" is going for each turn. They need to speak loudly enough to be heard, but moreover, they need to look at each other. Looking at each other is so basic a norm that as adults we barely think about it, but students in mathematics class may be used to looking at the teacher only—except when they are getting into trouble! Looking at people as they speak helps establish an atmosphere of respect where everyone seeks to understand each other.

Here is the discussion one teacher had with her students after Zip, Zap, Zop.

Ms. Haddad:	All right. Think about what we just did. Think about how does the game relate to having conversations in the classroom? How does that game relate to us and our classroom norms? Who would like to share a connection for us? Jadzia?
Jadzia:	It connects because when we were doing Zip, Zap, Zop, we were going back and forth, and it's like speaking but actually is a game. And when we were doing the conjecture, argument, we are also doing the same with ideas instead of using zip, zop, or zap.
Ms. Haddad:	Outstanding. So passing around the ideas just like we passed around zip, zap, and zop. Other connections, Kai?
Kai:	We was speaking loudly enough so everyone can hear.
Ms. Haddad:	Yes! Speak loudly enough so when you share your conjecture, everyone can hear and respond. Other connections from the voices we haven't heard from? Tamiko?
Tamiko:	We didn't always have to get it right.
Ms. Haddad:	Absolutely. Can you share a little bit more about that?
Tamiko:	We could mess it up, say the words in the wrong order, and we would just start again.
Ms. Haddad:	Was it okay?
Class:	Yes!
Ms. Haddad:	Yes, so when we're making our conjectures, it's okay if we are not right the first time as long as you try. Right? Usher?

(*Continued*)

(*Continued*)

Usher: Because like, if discussion takes time, one person's going at a time and nobody's talking over each other.

Ms. Haddad: Perfect.

In addition to the norms stated earlier, there are norms that are specific to each part of argumentation (Figure 1.4), and we provide warm-up games for establishing them in the chapter for each part.

ACCESS FOR ALL

SHARING MATHEMATICAL AUTHORITY

Argumentation has the potential to help students develop as powerful users and creators of mathematics, with your support, in a classroom community. This requires explicit attention to developing this power, especially for students who have been historically marginalized in mathematics classrooms because of their race, level of English fluency, gender, or learning differences. If your students haven't done much argumentation, they are likely to believe that the teacher or the textbook is the only source of mathematical truth. By helping all students make bold conjectures and take mathematics into their own hands, you help students learn to take on mathematical authority for themselves as opposed to being passive receivers of content (Boaler & Greeno, 2000).

The social nature of argumentation is key. Students' authority is not developed in isolation; students need to talk to and listen to each other in developing their justifications (Esmonde, 2009). And you have the opportunity to validate all contributions to an argument as important, even if they don't follow the path to a justification that you expected. You also play an important role in facilitating students' clear communication and ensuring that students come away from argumentation with clear mathematical ideas.

Conjecturing, justifying, and concluding each provide opportunity to remind students that they are in charge, as a class, of deciding what is true and what is false. Of course, you may need to step in with your own mathematical authority if students persist in supporting a false conjecture or using limited reasoning. Even when you present students with a conclusion different from theirs, you can recognize the authority of the class as a community of mathematicians interacting with the even larger group of mathematicians—that you as the teacher represent—making explicit connections between their thinking and that of mathematicians (Stinson, Jett, & Williams, 2013).

This shift in mathematical authority can help students develop positive identities as mathematics learners. A good resource on equity-based practices to encourage positive mathematical identities is *The Impact of Identity in K–8 Mathematics: Rethinking Equity-Based Practices* by Aguirre, Mayfield-Ingram, and Martin (2013).

The norms introduced in this and subsequent chapters can contribute to students' developing mathematical authority, as students parlay their experiences with the games into the realm of mathematical argumentation.

All the techniques in this book are designed to help you broaden participation in high-level mathematics. The more concrete success you observe from a variety of students, the greater your expectations of all students will be. And just as we recommend students work together in a classroom community, we encourage you to take on this work with colleagues; sharing your successes and challenges with your colleagues can be helpful to all of you.

Throughout the book, we provide advice for how to provide support for culturally diverse students, students with learning differences, and students whose first language is not English. This is indicated with an *Access for All* icon.

A little more about these students for whom we provide extra support: For those developing their English language skills, rather than describing students as having limited English proficiency, we emphasize that these students come to school with many resources, including their home languages. For brevity, we will refer to them as English Language Learners, or ELLs. Also, while teachers' experiences with students with learning differences include students receiving special education services, as in a vignette in Chapter 5, we by no means address the range of particular learning disabilities you may encounter, on which special education experts can provide advice.

GETTING STARTED WITH ARGUMENTATION

Although not every lesson you teach will include all four parts of the argumentation model, almost any lesson can and should include students' justifications. Below, we provide three steps for an easy entry into teaching argumentation, which you can implement while reading the book.

Step 1: Start by adding some justification to your existing lessons.

> Use small questions—try to use them as often as possible and students will become aware that they can explain and justify. For instance, you can ask, "How do you know that's true?"
>
> —Eighth-grade teacher

No matter what lesson you teach, students are making mathematical statements—even if they are just the results of calculations. You can start having your students justify their statements simply by asking, "How do we know it's true?" Get students to give their best reasons based on the mathematical representations they have developed or that you provide for them.

For example, if you are teaching a lesson on division of fractions, students can use visual fraction models and stories to justify whether $\frac{3}{4}$ divided by $\frac{1}{2}$ is $\frac{3}{8}$ or $\frac{6}{4}$ (or something else). Whatever their answer to the calculation, you can ask, "How do we know it is true?" Students may start by explaining the procedure they remember—or misremember. You may need to prompt students in using representations to justify why the procedure works, asking, "Can you draw a picture that shows why?" or "Can you tell a story that explains why?" (For a complete framework on students' thinking about fractions, see Empson & Levi, 2011.)

In another example, a class is working on simplifying algebraic expressions. The teacher asks students why they could apply the distributive property to an expression: "How do we know that $3(x + 2) = 3x + 6$, no matter what x is?"

She provides time for students to draw pictures, reason with numbers instead of variables, and translate into real-world situations. The arguments are not very many steps long, but they provide insight into why the distributive property holds, instead of students having to accept it as just the way it's done in mathematics class.

Clearly, "How do we know it is true?" isn't the only question to ask to stimulate argumentation. You can go to Chapter 4 on justifying to see how to use different forms of "why" questions for slightly different purposes and to Chapter 5 to read more about the use of representations in justifications.

Step 2: Teach a lesson based on one conjecture.

You can begin a lesson with a "controversial" conjecture—a mathematical statement on which students may have differing initial views. If you know where you want the class to head, you can start with a conjecture that you have designed to highlight a new concept or insight. Some teachers have found it useful to start the class with a false conjecture, particularly one that represents an early conception that students commonly hold. Often, starting with a false conjecture is a good way to bring beginning arguers into the conversation. As you teach the lesson, students' statements about the conjecture will bring to light the reasons they hold their conceptualizations. As other students disagree and say why, the students can gradually reshape their ideas.

For example, you could start a lesson on multiplication of signed numbers with the following conjecture:

> Whenever you multiply any two negative numbers, the result is always negative.

You can elicit justifications from those who agree and from those who disagree, and you can contrast the justifications. You will have to make sure that students have some basis from which to justify. For example, you could ask them to use a number line to look at patterns formed by taking products of a sequence of numbers from 3 to –3, each times 2, and then have them look at the patterns that must hold for the same sequence, each multiplied by –2. This justification relies also on the fact that operations on signed numbers must be consistent with each other.

Alternatively, you could ask students to consider a justification using the distributive property and the fact that the sum of a number and its opposite is zero by asking, "What are two ways to evaluate the expression $-4(-5 + 5)$? How can they help us find the product of –4 and –5?" Even when offering these possible representations, you can press students to provide the justification themselves.

The justification is that the expression helps us this way:

> By doing what is in the parentheses first, we get 0. By using the distributive property, we get $-4(-5) + [-4(5)]$. If we already know that negative times positive equals -20, then by substituting we get $0 = -4(-5) - 20$. So $-4(-5)$ must be positive 20.

Chapter 3 offers more advice about starting a lesson with a single conjecture.

Step 3: Teach a lesson that requires students to create their own conjectures.

As you move into Step 3, you'll have students engaged in all four parts of the argumentation model. Keep in mind, though, that not every lesson will contain every part of the model. You may, for example, have students generate cases and make conjectures one day, and choose conjectures to justify the next. To begin a set of lessons based on an activity such as those provided in this book, you may want to entertain a lot of conjectures, which will come up for justification in different parts of subsequent lessons.

Chapters 2 and 3 provide classroom activities in which you elicit conjectures from students through generating cases. Use the lesson planning advice we give in Chapter 8 to help you prepare for a sequence of argumentation lessons.

ARGUMENTATION LESSONS VERSUS ARGUMENTATION *IN* LESSONS

You may be wondering if you need special lessons to teach argumentation or if you can integrate argumentation into lessons and activities you are already using. Our answer: Both! To fully engage students in the process of argumentation implied by our model, you will need lessons that are devoted to argumentation, and they should always address important content standards as well. We provide many tasks for such lessons in this book. But in your everyday teaching, you will find that you can include justification in many lessons just by asking students to convince each other of their ideas. The chapters on justifying provide you with teaching moves for doing so.

WORKING TOGETHER

Simply reading this book is not sufficient for learning to teach argumentation improvisationally. You will need to try out new ideas in your own classroom gradually, so we provide advice along the way on how to get started. Even better, however, is reading the book together with your grade-level team or professional learning group and practicing your new teaching moves together before you try them out in the classroom. We present suggestions for what to do in such a group in each chapter's Working Together section; an important resource for us in developing these sections was ATLAS (National School Reform Faculty, 2014).

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(*Continued*)

Here's where to begin:

Exploration and discussion (30–45 min)

1. Convince yourselves: Why does the process for dividing fractions—the so-called invert and multiply—work?
    - Start with one example, say, $\frac{3}{4}$ divided by $\frac{1}{3}$ is $\frac{9}{4}$, and use a story or diagram to justify it.
    - Then using the example, discuss with your colleagues why the process always works.

Wrap-up and assignment (15 min)

2. Write down two or three questions to ask your students over the next few days. At the next meeting, report to the group about how the questioning went in your classroom. What worked? What was frustrating? (10 min)

3. During the week, commit to asking at least twice in one lesson, "How do we know it is true?" Briefly write down how you plan to respond to the justifications that students give and guide them toward making stronger justifications. (5 min)

NOTES