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Making 7—The Lesson

(Grades PreK–K)

For much of the 20th century, learning single-digit addition and subtraction has been synonymous with memorizing math facts. The predominant learning theory supporting this view was *stimulus-response:* A card with "3 + 4" flashed before children's eyes was the stimulus, and the memorized, isolated answer of "7" was the expected response. Despite all we now know about how the brain learns mathematics (Dehaene, 1997; Sousa, 2008), in many schools, memorizing mindless math facts is still stressed by some parents . . . and teachers.

Making 7 explores alternatives to rote memorization and fosters sense making of single-digit addition combinations. The lesson described below took place in the classroom of Ms. Brady, an extraordinary kindergarten teacher. Ms. Brady had split up her class into two groups: Group 1 was her "regular students," Group 2 her "high flyers." I worked with each group separately using different explorations into the same mathematics. The lessons were filmed and discussions transcribed.

BOX 1.1 REFLECT . . .

Suppose a parent told you, "My daughter Ping already knows her addition facts for making 10. I'm afraid she's not being challenged in math." What would you devise for Ping and other students like her?

GROUP 1

Setup

Ms. Brady's ever-growing love for and knowledge of mathematics were contagious. The children's excitement was palpable. Seated in a circle on the colorful learning rug, they were eagerly waiting for Dr. Monica. The manipulatives requested—two large dice, unifix cubes sorted by color, and a white board—were ready, as was a beautiful drawing by Ms. Brady on a magnetic green board. A large, leafless tree was drawn on the right of the board and a small dense bush on the left. The artwork had intrigued the children since their arrival that morning, but as requested, their teacher had kept its purpose a secret. I like adding suspense and theatrics to my math classes. It comes from my theatre experience.

Dr. M: You know I'm a mathemagician, right?

Some (*politely*): Yes.

Dr. M: Would you like me to start our lesson this morning with a trick from my *MathMagic* show?

All (*excitedly*): Yes!/Yeah!

Dr. M (*picking up the large dice, one in each hand*): Before I perform my trick, I have a question for you: What are these?

Some kids called out "cubes"; others said "dice." I addressed both:

Dr. M: Yes, a cube is the name of their shape. But dice is also correct. Why?

Chloe: Because they have numbers on them.

Dr. M: Absolutely! How many *different* numbers do you think are on these dice?

Many hands went up. Answers included a couple of *four*s, some *five*s, but mostly *six*es, so we went with *six.*

Dr. M: So can anyone tell me what the six numbers *are?*

Silence fell upon the classroom. My question was too ambitious. So I modified it:

Dr. M: What's the smallest number?

Many children (*in unison*): One!

Dr. M: What's the biggest number?

Many children (*again in unison*): Six!

Holding up just one die and rotating it to reveal the cube faces in random order, I asked the students to call out the number of dots I was pointing to. They correctly identified the dot numbers in random order, then in ascending order calling out, "one, two, three, four, five, six." Revisiting their first answer, "cubes," I asked one more question, nudging them toward the one-to-one correspondence between the six dot numbers on a die and the six faces of a cube:

Dr. M (*placing my entire palm against one face*): This is called a face of the cube. Some of you called it a side. How many faces, or *flat* "sides," are on this cube?

Once again, silence. So I placed the red die in the middle of the floor for all to observe, as I retreated from the center to take a seat among the children, in the circle.

Jacob: Six.

Tyler: No, five!

Elizabeth: It's six because there's one on the bottom.

One girl said "four." Keeping a neutral face so the students would persevere, I broke down the number of faces into three parts, calling them top, sides, and bottom. Students successfully called out the corresponding numbers for these three subgroups, namely, 1 (for the top), 4 (for the lateral sides), and 1 (for the bottom). We then added them together in two stages as I simultaneously gestured to indicate each addend of the sum: "How much is one plus four?" I asked. "Five." "How much is five plus one?" "Six."

Then, before I could utter my closing question, an amazing thing happened: Juliana leaped from her spot to the center of the circle, grabbed the cube, and said, "I know! There's like . . . um . . . six sides on it and each side has a number. So there's six numbers!" This was a significant insight into the one-to-one correspondence I was probing into. Notice Juliana's answer: She had connected the number of "sides" to the number of dot-numbers, so much so that she actually answered "So there's six *numbers*" to my question about "How many *faces?*" Thrilled, I exclaimed, "You just got what I was hoping you'd figure out!" To reinforce this connection that only a few students had grasped, we concluded by singing, "*Six numbers for six sides, six sides for six numbers.*"

Discussion

1. The Math-Magic Trick

We were all ready for the magic trick. I rolled the two large dice into the middle of the circle and had students identify the dot numbers "on top" (Figure 1.1). Five for the red die and four for the black one they said, accurately. Mesmerizing the class with a trembling gravity in my voice, I intoned, "Abracadabra, using my mathemagician eyes, I'm looking through these dice and I see that under the four, there is a three; and under the five, there is a two."

FIGURE 1.1 Math-magic trick: Seeing through the dice

A girl's voice (*breaking the silence*): On the bottom?

Dr. M: (*clarifying*): Yes, *on the bottom* of the black die there's a three (*I picked up the black die to reveal the* 3), and *on the bottom* of the red die there's a two (*I picked up the red die to reveal the* 2).

Noah: I think you remembered them.

Emma: I don't really think you're looking through the cubes . . . I think . . . I think that . . .

Sadie (*interrupting, and pointing to the 4*): . . . that you see around the cube that there's not that number.

Emma (*pursuing her own idea, also pointing to the 4*): So you know it's a three on the bottom because it's the opposite of four . . . kinda.

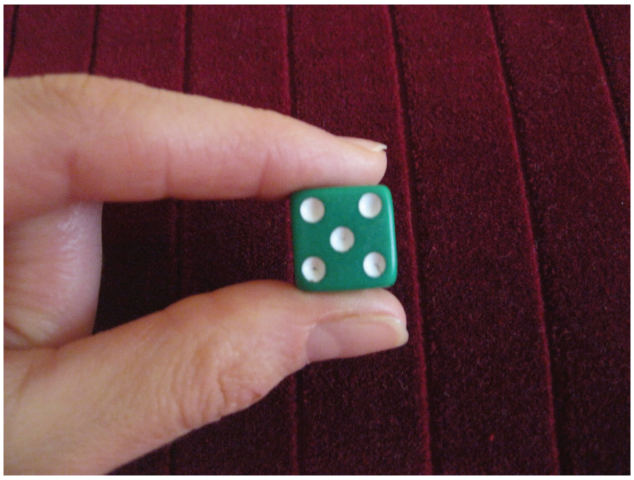
Sadie (*insisting on theory, and clarifying*): No, you look around it [the 4] and there's no three around, so you know on the bottom there's a three.

Emma was onto something. I encouraged her to think further about her theory of "opposites." But to the class, Sadie's theory was convincing: When asked how I knew there was a 2 under the red 5, the class had adopted her idea. I acknowledged that looking around to find the missing number was a nice explanation, but that there was more math to my magic. To convince them, I turned my back and assigned two quieter students to roll the dice and then call out the numbers on top. Without looking, I had to identify the numbers on the bottom. They obliged me. Ethan called out a "one" for the red die and Abigail a "three" for the black die. Nicely, two *new* numbers were rolled.

"Ethan, on the bottom of the one there's a six; please turn it over to show the class," I said. Sounds of amazement filled the room. After a few seconds, and still with my back turned, I continued, "Now Abigail, on the bottom of the three, there's a four. Your turn to check if my math-magic worked." The children were dumbfounded. I turned back to join the circle of smiling yet puzzled faces.

2. A "Put-Together" or "Combine" Addition Situation

FIGURE 1.2 Holding the die between thumb and index finger



"Let's try to figure out the mystery; then you all can be mathemagicians too," I began. Distributing a small colored die to each student, I demonstrated how to "hold onto it with just two fingers: your index finger and your thumb" (Figure 1.2). Next, I asked each student to guess, without peeking, "How many dots you have hiding *altogether* under *both* fingers?" Guesses ranged from one to ten. "Next I want you to look under your fingers and count *all* the dots you see hiding under your *two* fingers—the dots under top finger *and* the bottom finger." With a few exceptions of students who just counted the dots *on top,* as I went around the circle, one student after another called out "seven." They successfully added dots of opposite sides. The repeating sum of "seven" prompted some conjectures:

Juliana: Oh I know I know; all the sides add up to seven.

Dr. M: *All* the sides?

Juliana: Yeah!

Dr. M: You mean if I add up the dots on *this* side, and *this* side, and *this* side, and . . . (*pointing at all six faces, one by one*), I'll get seven?

Juliana (*with a hint of doubt in her eyes*): Maybe?

Alicia (*interrupting*): Every *opposite* number makes seven!

Dr. M: What do you think Juliana?

Juliana acquiesced that Alicia made sense: It was not *all* sides but rather *opposite* sides that added up to 7.

3. Partners of 7

Shifting the discussion back to the individual addends, I said, "Now I will write down your names and record the numbers you have under your two fingers." To this end, I proceeded to make a table (Table 1.1). As I turned to the white board, a student blurted out, "I kinda lost my numbers." "Not a problem," I replied. "Just hold the die with two fingers and read me those numbers." While I was writing Chris's sum (see Line 2 of Table 1.1), a voice exclaimed, vehemently, "That's the same thing!"

Dr. M (*pausing to inquire*) What do you all think; am I writing the same thing for Chris and Kyle?

Kyle (*assuredly*): It's different but it makes the same thing.

Dr. M: That's an interesting way of putting it. Can someone explain what's *different* and what's the *same?*

Some students had not been attentive, so Kyle repeated his statement. Ishan clarified as follows:

Ishan: Five was first but now it's second but it still makes seven.

Dr. M: I think I get it: The *order* of five and two is different, but in both cases the numbers add to seven? (*Ishan agreed.*) Does that make sense to everyone?

The class seemed convinced. I filled in the table with a few more answers. This time, there was no objection to Ethan's "six and one" or Chloe's "four and three."

TABLE 1.1 Students' dot numbers hiding under their two fingers

Name	Top		Bottom		Total
Kyle	5	and	2	⇒	7
Chris	2	and	5	⇒	7
Elizabeth	1	and	6	⇒	7
Tyler	3	and	4	⇒	7
Ethan	6	and	1	⇒	7
Chloe	4	and	3	⇒	7

We continued around the circle, inserting additional names into the *Name* column, grouping all student names into six subgroups, corresponding to the 6 "making-7" combinations shown in the table. (Students may also know these pairs as "partners" of 7.) The students were convinced that our table had captured all possibilities. It also reinforced the fact that only pairs of dot numbers on *two, opposite* "sides" of a die add up to 7.

Delving Deeper

4. Part-Part-Whole

"Now *you* can be the mathemagicians. Let's roll the large dice and see if you can figure out—without looking—the numbers at the bottom," I cajoled. Realizing on the spot that rolling two numbers may distract them from focusing on the hidden number under any one die, I changed plans in midcourse: "Actually, let's begin with just *one* die." I rolled a 3, which was immediately identified as a "three." "Raise your hand if you think you know the number hiding underneath," I said. A couple of students appeared to know right away as they confidently raised their hands high. Some looked at the white board for help. Others crawled around to see the lateral faces, presumably intent on applying Sadie's missing-number theory. Yet others had no clue. Modeling the situation by writing a familiar looking number sentence (Table 1.2) on another white board, I asked, "If we know the top has *three* dots, how many dots must be on the bottom if the total number of dots is *seven?*"

TABLE 1.2 Modeling the math-magic trick with 3 dots on top

Top		Bottom		Total
3	and	?	⇒	7

TABLE 1.3a Part-part-whole table

Part 1	Part 1
Whole	

TABLE 1.3b Part-part-whole table to record daily attendance

Absent	Present
Total Class Size	

I also offered an alternate representation in the form of a part-part-whole table (Table 1.3a). The students were familiar with such tables as they regularly used them in class, for instance to record daily attendance (Table 1.3b). I was hoping that their familiarity with this tabular representation would help them sense an analogy between two situations that seemed different to them but are mathematically equivalent (Tables 1.3b and 1.3c).

I noticed Ayiana, who had been quiet but intent throughout the lesson, holding up three fingers on the right hand. On the left, she gradually held up one finger, then two, then three, and finally four fingers. When she was done, she raised her hand timidly.

Dr. M: Ayiana, what were you trying to figure out? You had three fingers on one hand; how many were on the other hand?

Ayiana: Four.

Dr. M (*delighted*): Why four?

Ayiana: Because it has to be seven.

Dr. M (*delighted*): Nice! And remind us *what* always "has to be seven" on a die?

Ayiana (*as if reciting a refrain that was playing in her head*): Opposite sides.

Dr. M: Absolutely! Did you all hear what Ayiana said? (*reinforcing the learning for the whole class*): The numbers on *opposite* sides of a die always add up to *seven.*

TABLE 1.3c Part-part-whole table to model math magic trick

Top 3	Bottom ?
Total 7	

Ayiana appeared to have used the counting-up method to find the missing addend. She solved the number sentence "3 plus WHAT makes 7," or the equation 3 + ? = 7. She began with three fingers on her right hand then must have counted, in her head, "four, five, six, seven" on her left, keeping track that "four" was **1**, "five" was **2**, "six" was **3**, and "seven" was **4**. This count-on method to find the unknown addend is practiced by first- and second-graders, but some children like Ayiana are able to grasp it earlier. For most students in Ms. Brady's Group 1, however, this was a challenging task as expected.

We played being mathemagicians a couple more times, still with just one die. I concluded this part of the lessons by stating, "Boys and girls, the beauty behind this trick is the mathematics: Your secret when performing the trick is knowing that, no matter what number you roll, *top plus bottom always make seven!*" While young children want to believe in magic—and many do, it's important to unpack the mathematics and stress that the "magic" behind this trick is none other than the beauty and power of mathematics: The three ways of making 7 (with two addends, using 1 through 6 only once each) are matched up with the three pairs of opposite faces in a cube.

5. Coming Full Circle

To bring closure to this lesson, I revisited the symbolic notation for partners of 7 and drew a final connection between these addition combinations and the geometric structure of the cube, which we had discussed at the start of class.

Formalizing Notation. "When joining two numbers together," I inquired (pointing at the "and" in the sentence "5 and 2 ⇒ 7" from Table 1.1), "What sign should we put between them to . . ."

"Plus!" A chorus of children interrupted. I also asked for a mathematical symbol for "⇒" or "makes all together." The equals sign was less popular than the plus sign, but still familiar to the class. Together we made a new table (Table 1.4), rewriting the number sentences in Table 1.1 with symbols + and =. Reiterating Kyle's phrase of "*different* but makes the *same*" (e.g., 5 + 2 and 2 + 5 are different, but the sums are the same), we identified the other two pairs that "go together," namely 1 + 6 and 6 + 1, and 3 + 4 and 4 + 3. Savannah suggested to "put these together like the ones with *five* and *two*," which we did.

TABLE 1.4 Equations modeling the sums of dot numbers on opposite faces of a die

Top	+	Bottom	=	7
5	+	2	=	7
2	+	5	=	7
1	+	6	=	7
6	+	1	=	7
3	+	4	=	7
4	+	3	=	7

While primary students are only beginning to make sense of

1. Symbols such as + and =,
2. Expressions such as 5 + 2, and
3. Equations such as 5 + 2 = 7,

their understanding of these notions develops as they come to see them used as models of real-world situations they experience. In addition to the more commonly used verbal, kinesthetic, concrete-manipulative, and pictorial representations of relationships among numbers—and later among quantities, I encourage using numerical, symbolic, and graphical representations of real situations whenever possible.

BOX 1.2 RULE . . .

The ancient Greeks named 3-D solids by the number of ways they could “sit” on a horizontal surface. Therefore, to figure out the name of a polyhedron:

1. Count the number of faces,
2. Find the Greek root for that number, and
3. Combine it with the word hedron (“seat” ⟶ face).

For example, a hexahedron (or a cube) has six faces, and a tetrahedron has four faces.

Why Are There Six?

As a final inquiry into this mathematical exploration that began with a simple math-magic trick, I proposed, “Let’s count how many different ways a die can land on the floor when rolled.” We all counted together and found six. “Does it make sense that a cube can land in six different ways? Why *six?*” I heard some mumbling, but it was time for closure and connections. Placing my palm flat on one face of the cube, I asked, “How many flat sides did we discover a cube has?” “Six,” answered La Taya. “So if you think of a flat side as a seat, how many ways can a cube sit on the floor?” “Six!” they said again, this time with amusement. (*Note:* The term for a 3-D geometric solid with flat faces and straight edges is *polyhedron*. In Greek, *poly* means *many*, and *hedron* means an amphitheater *seat*.) Their final task was to reflect on the connection between the number of equations in Table 1.4 and the number of faces on a die. They would discuss their findings the following day.

When teaching a lesson, much is exposure, stimulation, and thought provocation. I had no illusion that all students would remember everything. Far from it. But if I plant thinking seeds, students will go back and reflect. They’ll remember that a cube has six faces/seats (geometry), that a die has six numbers on it (number), and that there are six different ways to add top and bottom numbers on a die when it’s rolled (arithmetic/algebra). One day, these ideas will all converge and . . .”Eureka!”

BOX 1.3 RESEARCH . . .

Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.

Jan A. Van de Walle & LouAnn H. Lovin (2006)

GROUP 2

Setup

The high flyers, as Ms. Brady sometimes called them, were waiting at the door. As soon as the first group had left the room, they entered. They were five in this group, two girls and three boys. We formed a small circle on the learning rug. "Do we still like math?" I began. All five raised their hands enthusiastically. "I love it!" said one child. "It's like my favorite thingy!" said another, making everyone laugh. Piggybacking on this exclamation, "It's my favorite thing in the world," voiced a third. I told them I wrote a story especially for them, to which Jag responded, "I'm the best at math problems." Coming from an older child, this self-praise would have been boastful. But here it was more like an accurate assessment of self. Plus, his peers were not offended; with their silence, they acquiesced. I nevertheless whispered, "It's nicer to let *others* say that about you." In any case, the positive disposition toward mathematics was palpable. The purpose behind Ms. Brady's artwork on the green board was soon to be revealed.

Discussion

1. The Bird Story

Liana lives in Virginia, in a town called Machipongo. She loves to play in her back yard. She especially likes climbing the tallest tree and playing hide-and-seek. Her favorite place to hide is in the nearby bush. No one ever finds her in there. Every year, as winter approaches, she watches the birds fly over her house on their way to warmer places. But this year, something special happened. When the time came, Liana found seven, little, purple birds—with small black beaks, and small black eyes, and mellow yellow bellies—taking a break in her back yard. Some were perched up high in her favorite tree, and some were perched down low in her favorite bush.

My Question. Try to imagine these seven, little, purple birds—with small black beaks, and small black eyes, and mellow yellow bellies landing in Liana's back yard. How many do you think perched up high in the tree, and how many down low in the bush?

I had cut out two-dimensional bird shapes from purple tag board, colored the beaks and eyes in black and used round, yellow magnets to pin them to the board. "I'll ask each of you, one by one, for a possible answer, and I want you to remember it, OK?" Eager to share their ideas, they patiently waited their turns.

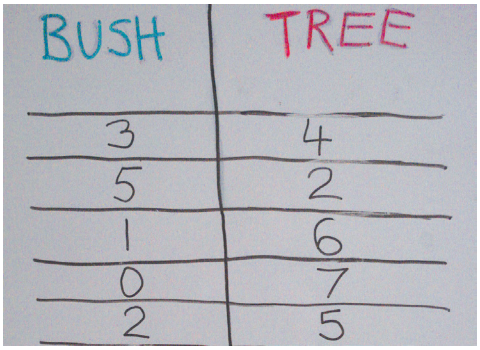
2. A "Take-Apart" Addition/Subtraction Situation

FIGURE 1.3 Jag's idea: 4 birds in the tree and 3 birds in the bush

"Four liked the tree so much, four landed on the tree, and seven . . . um, I mean three, ended on the bush," began Jag. We modeled the first possibility as depicted in Figure 1.3. "I got an idea," said Carlos. "What's your idea?" I echoed. "I think the tree has two and the bush has five." He went to the board and, accordingly, shifted 2 birds from the tree to the bush. Diana was next: "One and six" she declared. "Where is the *one?*" I probed. "In the bush!" she answered, shifting all the birds, except 1, back to the tree. Then she moved 1 of the 6 birds to the edge of a branch, isolated from the others, saying, "That way this one can make her own nest." Po Yew took her peers by surprise with her original suggestion: "All seven in the tree." The others had surmised from the question's wording that there had to be some birds in *each* location. Lastly, Kane rearranged the birds hesitantly, worried that he may be duplicating a previous answer, "I don't know if anybody had this one," he whimpered. His fear was confirmed. "Hey that's the same one I did!" cried Jag, namely 3 in the bush and 4 in the tree. "Just put one bird back [on the tree]," he added. Now there were 2 on the bush and 5 in the tree. A voice exclaimed, "Carlos already did that one!" to which Jag, with his phenomenal memory, contested "No, no, um, he did five on the bush and two in the tree. It's the *opposite.*"

3. Partners of 7

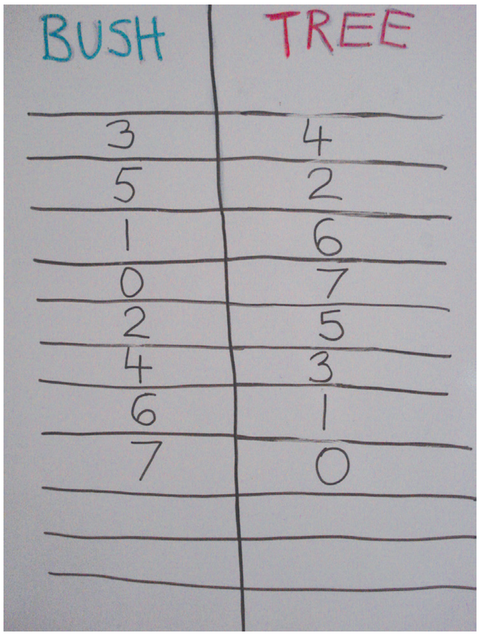
FIGURE 1.4 T-table representing students' answers



After one round of suggestions, I asked, "How many different answers do we have so far?" "Five," they replied unequivocally, without any visual record. "How do you know we have five?" I continued. "Because we're five!" said Po Yew, confirming an understanding of the one-to-one correspondence between number of answers and number of students. To help us keep track of old and new combinations, Kane proposed that I "write them down on the white board." Taking his suggestion, I constructed a T-table with *BUSH* on the left and *TREE* on the right, like in the drawing, and recorded their answers as the students dictated them to me. Everyone remembered his or her partners of 7 (Figure 1.4). When Po Yew called out her answer (0 → bush, 7 → tree), she correctly articulated "*Zero* in the bush and seven in the tree." I acknowledged a correct use of zero by remarking, "Oh, I like that: Zero means none in the bush, right?" She nodded. Incidentally, she was also the one who spelled tree for me: "*T, R, E, E.*"

"Now listen carefully to the next question: Could we come up with *more* possibilities for 7 birds landing in two places, by looking at our chart?" They all voted "yes," so we added more empty rows in anticipation (more than necessary, on purpose). Diana was first to add the ordered pair (4, 3).[1] Jag, author of the first entry (3, 4), reacted out loud, "Yeah, that's the opposite of mine, I'm three-four, you're four-three." The group agreed that (4, 3) and (3, 4) offered two different answers to our problem. I shared with them that "some people call these turn-around facts."[2] They liked the expression. Kane added the next entry, (6, 1), claiming it was "the turn-around of Diana's answer." "Can I go up next?" asked the discreet, pony-tailed girl. With a smile on her face, Po Yew followed with 7 in the BUSH column and 0 in the TREE column, swiftly adding—as if to beat the others to it, "I turned mine around." Now our T-table represented eight possible ways for 7 birds to perch in two places (Figure 1.5).

FIGURE 1.5 **Table of student-produced possible ways for 7 birds to perch in two places**



Delving Deeper

4. A Shift in Discourse

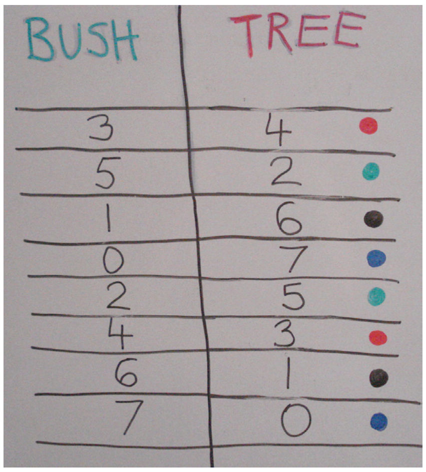
No sooner had Po Yew entered the last row of numbers in the T-table than Carlos spoke up, "I don't have any more!" He was responding to Jag's coaxing to go up next with another solution. I asked for clarification, "What do you mean Carlos? Did you say you don't think there are any more ways?" Jag, who was sitting next to Carlos, interjected, "I *know* there's more." Unwilling to contradict Jag, the boy with the reputed problem-solving ability, or perhaps influenced by his conjecture, Carlos compromised, "I think there's more but, um, I don't have another idea." "But you first said there *aren't* any more, right?" I added, to draw attention to his initial thought. I summarized the two conjectures, "So, Jag, *you* think there *are* more ways, and *you*, CARLOS, think there are *no* more ways."

How Can We Know?

The duo was onto something interesting. The class was split: Two agreed with Carlos and three with Jag. How could we figure out whose conjecture was correct? We had a new conundrum. "We need to know," I insisted. We began by counting the ordered pairs the group had come up with. We found eight. Kane had an idea, "Maybe some turn-arounds are missing." "Great idea! Let's check," I replied. Taking the black marker, I drew two big dots beside (1, 6) and (6, 1), thus modeling a strategy for matching turn-around pairs. Three students followed suit, drawing red, green, and blue dots respectively beside the other three pairs. This strategy confirmed that each *BUSH-TREE* ordered pair had a turn-around partner. The eight combinations were now seen in a new light: four groups of two, labeled as two black, two red, two blue, and two green (Figure 1.6).

The class discussion had shifted from partners of 7 to partners of equations representing relations among the same set of numbers (e.g., 7 = 3 + 4 and 7 = 4 + 3 are turn-around partners, because they

FIGURE 1.6 **The eight answers, color-coded—two by two—with black, red, blue, and green dots**



both express relationships among the same set of numbers, namely {3, 4, 7}[3]). Indeed, each table row represents a relation among a set of three numbers. For instance, Row 2 represents a relation among the numbers in the set {2, 5, 7}; namely, it conveys that 7 = 5 + 2.

The color pattern seemed gratifying to the children. The fact that every combination had a partner was reassuring, "just like even numbers," someone noticed. Moreover, since the possible solutions were now paired two by two, and since no one could think of a new solution, the group was strongly leaning toward Carlos's conjecture: They were convinced there were no other ways for seven purple birds to perch in a bush and a tree.

5. Bringing Closure

How Can We Be Sure?

Proof is what distinguishes mathematics from other subjects. Beliefs or convictions are not enough. In mathematics, we are asked to prove our conjectures, ideas, and answers. Ms. Brady's Group 2 students found the questions, "How can we be sure?" and "How can you prove it?" difficult but not overwhelming, as they were regularly asked to explain their answers (logical reasoning) and to convince others of what they held to be true (mathematical proof).

TABLE 1.5 The eight answers reordered, with the BUSH-numbers in ascending order

BUSH	TREE
0	7
1	6
2	5
3	4
4	3
5	2
6	1
7	0

As the final questions were challenging, and time was almost up, explicit guidance was needed: "Look at the first column," I said. "What do you notice?" Jag observed that all numbers "from one to seven, *and* zero" were present. "Can you think of another way we could order these numbers?" I continued. "Like when we count?" wondered Po Yew, emerging from deep reflection. Why not! After some talk about where zero should go, the students dictated the pairs to me once again, ordering the *BUSH*-numbers "like when we count," from smallest to greatest. They had no idea that the final *BUSH-TREE* chart thus created (Table 1.5) held the key to our "proof."

Suddenly, it all made sense. The simple rearrangement of the same eight combinations was thought provoking. The pattern's regularity, in each column, was eye opening. "Hey, one goes down, one goes up," commented one child. "They're all there so nothing's missing," observed another. We pursued this comment further. Indeed, the children could see that the uninterrupted sequence of numbers, from 0 to 7, in *both* columns, represented *all* the possible ways that the 7 birds could have perched in each location. That reasoning provided an acceptable proof that all possibilities had been accounted for.

BOX 1.4 READ . . .

> The ability to reason systematically and carefully develops when students are encouraged to make conjectures, are given time to search for evidence to prove or disprove them, and are expected to explain and justify their ideas.
>
> National Council of Teachers of Mathematics (2000)

The icing on the cake came from Diana, who, after drifting off into a momentary slumber, contended "There're eight ways 'cause the zero on top makes eight." Hers was an unsolicited answer to the question I was barred from asking by lack of time, namely, "Why do you think there are *eight* ways of making seven?" Diana planted a seed for a follow-up exploration. I commended the group for their attention, participation, and great ideas. Then, off they went to recess, empowered by a convincing "proof" that *all* possible answers to the mellow-yellow-belly bird problem had been found. A gratifying closure to a multifaceted discussion.

Next Steps

For possible next steps to this lesson, Ms. Brady's follow-up ideas, which she explored with her students herself, offer some insightful suggestions. They are related below.

Ms. Brady had observed the entire lesson. She never ceases to be amazed by her students' potential. She is not one who belabors what her students *can't* do but rather exalts what they *can* do. She took notes and planned to view the videotape of the lesson. We discussed ways to revisit and extend the lesson. In particular, she wanted to explore Liana's bird story with the *whole* class. She also wanted to delve deeper into Diana's observation, making it more explicit for all the children and generalizing the finding for other numbers of birds.

On my following visit, Ms. Brady shared the highlights of a follow-up lesson in which students were asked to construct all possible, different, two-color trains with seven unifix cubes. But they had to abide by one *constraint:* "In two-color cube trains, same-color cubes must be grouped together," she told the class. "They couldn't be mixed up."

Ms. Brady had jotted down, in her journal, instances of children's thinking that had impressed her:

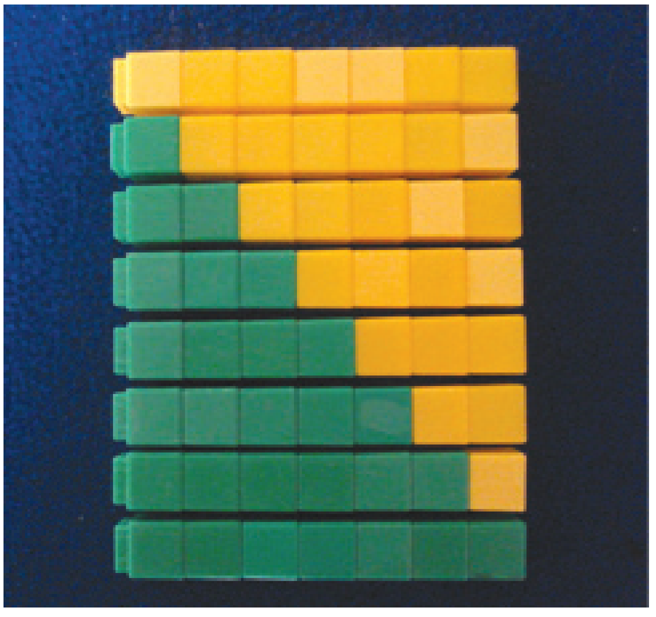
> Kane said out loud that the 7 cubes reminded him of the 7 birds. He didn't elaborate on his thinking, but his words sparked further analogies between the two problems.

Devika, upon hearing Kane's observation, chose yellow cubes to represent birds in the "dry" leafless tree, and green cubes to represent birds in the dense, "green" bush.

Po Yew and Devika, working together, found all eight possible two-color trains. Placing their eight trains in order, as shown in Figure 1.7, and pointing to the green cubes from top to bottom, they commented that "the green staircase is like the 0, 1, 2, 3, 4, 5, 6, 7 in the BUSH column" (referring to Table 1.5).

Many students, however, were not able to abide by the constraint of keeping cubes of a

FIGURE 1.7 Devika and Po Yew's layout of their eight solutions to the two-color train problem with 7 cubes



same color together. Some students set their creativity free and created a variety of artistic renderings (Figure 1.8).

On different days, the children were asked to repeat the same two-color train problem with 3 cubes, then with 5 cubes, and finally with 10 cubes. Five students observed and expressed a generalization from the combined explorations: "There's always one more [train than the number of cubes per train] because of the zero," as Jag put it. This is true only if the constraint of grouping cubes of the same color together is respected:

FIGURE 1.8 Kindergarten students showing off their two-color, artistic trains with 7 unifix cubes

Photo by Sarah Coste

3 cubes → 4 two-color trains

4 cubes → 5 two-color trains

5 cubes → 6 two-color trains

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10 cubes → 11 two-color trains

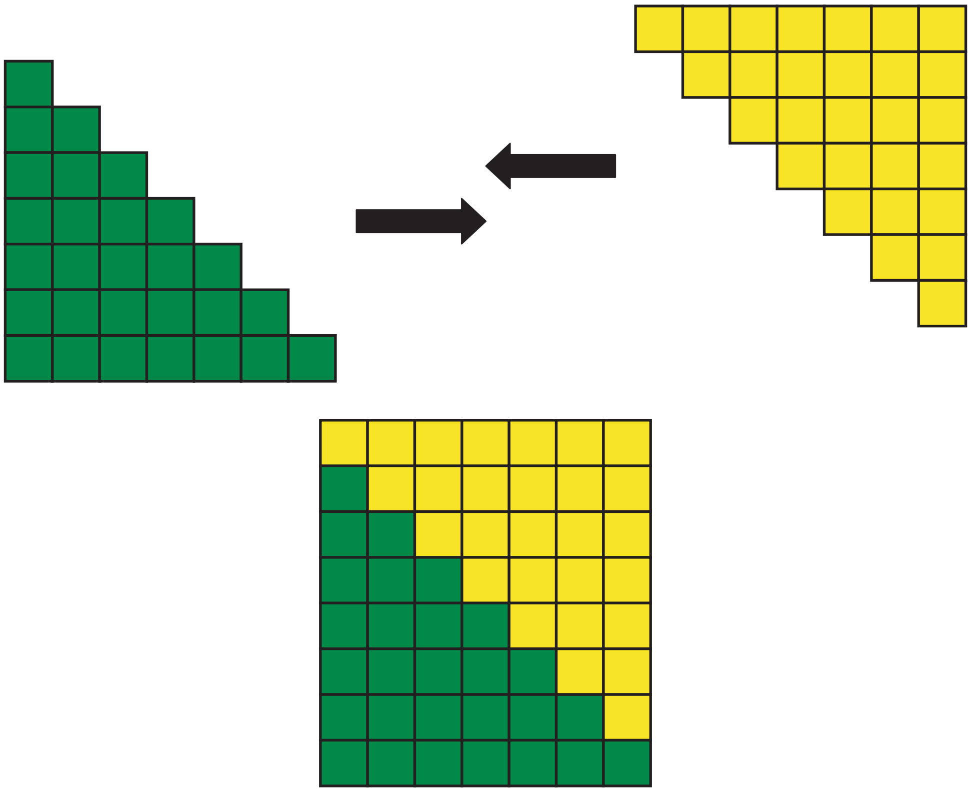
n cubes → $n + 1$ two-color trains

After listening to Devika and Po Yew and observing the green staircase they constructed with their eight two-color trains, Lloyd became fascinated by the staircase representation of consecutive counting numbers. He found them "cool," and proceeded to build staircases with multilink cubes, from 1 to 10 stairs high and beyond. Ms. Brady had him share his concept with the class. Others quickly followed suit.

The student-coined "staircases" helped Ms. Brady see the problem through a new lens: "Actually, to find all the different [two-color, 7-cube] trains, you just need to build two 1-through-7 staircases, and then interlock them together—one right-side up and the other upside down—and make a 7-by-8 rectangle, like this (pointing to two single-colored staircases Lloyd had created, Figure 1.9). That's why there're eight ways." She planned to wait a few days before revisiting Liana's bird problem, but this time with 10 birds. She would propose using 10 unifix cubes to represent the birds, and two colors to represent the locations. She was curious to see if students would transfer their "staircase discovery" from the cube-train problem and apply it to the bird problem. How many, she wondered, would build two 1-through-10 staircases, interlock them into a 10-by-11 rectangle, and find 11 solutions? "Probably Lloyd, Jag, Devika, and Po Yew for sure," she predicted.

Such aha moments—experienced by students and teachers, are the joy and beauty of mathematics. Ms. Brady's hope is that *all* teachers orchestrate their instructional activities in such ways that *all* students experience the joy and beauty of mathematics *every day.*

FIGURE 1.9 **Two 1-7 staircases combine to produce a 7-by-8 rectangle. The 8 rows of 7 cubes represent the eight solutions to the bird and train problems.**



Notes

1. In mathematics, (x, y) denotes an ordered pair of numbers from a T-table. It is called "ordered" because the order counts. For example, in this situation, (4, 3) is different from (3, 4).

2. Their school used Everyday Mathematics for grades preK–5. Everyday Mathematics calls expressions such as 3 + 4 = 7 and 4 + 3 = 7 *turn-around facts.* This phrase was therefore in concert with the terminology the students were going to learn in the coming years.

3. Braces are used to denote sets.