Chapter 10

DEVELOPING STUDENTS' GEOMETRIC THINKING

The teaching of geometry has a long, rich history. One of the most important works in the discipline, **Euclid's *Elements***, dates back to approximately 300 B.C. In it, Euclid compiled and extended the work of his predecessors. Boyer and Merzbach (1989) characterized the *Elements* as the most influential textbook in history, and speculated that it may be second only to the Bible in terms of number of editions published. Euclid's *Elements* reflects the Greek commitment to establishing and teaching a deductive system of thought in mathematics. Studying such a system can be contrasted with learning geometry for more practical purposes, such as determining measurements. Even though the Greek word *geometria* is rooted in *geo*, meaning "earth," and "metron," meaning measurement (Rubenstein & Schwartz, 2000), the *Elements* does not emphasize the practical measurement aspect. Between the time *Elements* was written and the establishment of modern schools, the question of whether to emphasize the theoretical or practical aspects of geometry has persisted (Stamper, 1906).

The first attempts to bring geometry to schools in the United States focused on the study of deductive reasoning. Geometry made its way into the high school curriculum in the 1840s as colleges added it to their admissions requirements (Herbst, 2002). During this period, students were expected to memorize proofs from textbooks with expositions of Euclidean geometry. It was not until near the beginning of the 20th century that having students produce their own original geometric proofs became a widespread practice. The move toward having students produce original proofs was largely catalyzed by Bull Wentworth's textbook series, which became known for its inclusion of "originals" and came to dominate the textbook market (Donoghue, 2003).

At the outset of the 21st century, new questions about middle and high school geometry exist. Two major questions that have emerged in the past few decades are (1) how does geometric thinking develop? and (2) what role should dynamic software packages play in teaching and learning geometry? Researchers have devoted a great deal of attention to each of these questions. Their findings have important implications for teachers and, accordingly, will be discussed in detail in this chapter.

WHAT IS GEOMETRIC THINKING?

To understand how to support the development of students' geometric thinking, it can be helpful to begin by considering its nature. Geometric thinking can be examined from two different perspectives: (1) the thinking of mathematicians as they are engaged in doing geometry (hereafter referred to as **normative geometric thinking**), and (2) the thinking exhibited by students as they learn geometry. Since the ultimate goal of instruction should be to help students engage in normative geometric thinking, it is important to understand the maturation process leading to its attainment. To provide perspective on normative thinking, Cuoco, Goldenberg, and Mark's (1996) **geometric habits of mind** are considered below. Then, work inspired by the research of Dina and Pierre van Hiele (van Heile, 1986) is discussed as a means of understanding students' thinking while learning geometry.

Normative Geometric Thinking: Habits of Mind

Geometers exhibit a variety of habits of mind in carrying out geometric investigations. These include using proportional reasoning, using several languages at once, using a single language for everything, reasoning about systems, studying change and invariance, and analyzing shapes (Cuoco et al., 1996). As a starting point for understanding normative geometric thinking, each habit of mind is considered and illustrated below.

Proportional reasoning is a vital element in reasoning about things such as vectors, fractals, and theorems about planar objects. As an illustration of the central role of proportional reasoning in geometry, consider the diagram shown in Figure 10.1.

Suppose we know that circle C in Figure 10.1 has a radius of 5, $\overline{AB}$ is a diameter, $\overline{BD}$ is a tangent, and $m\overline{BD} = 7$. From that information, it is possible to determine $m\overline{AE}$. First, note that $m\overline{AB} = 10$ because it is twice the length of the radius. In addition, $m\angle ABD = 90°$ because the intersection of a tangent line to a circle and its diameter

**Figure 10.1** Diagram for a geometric proportional reasoning problem.
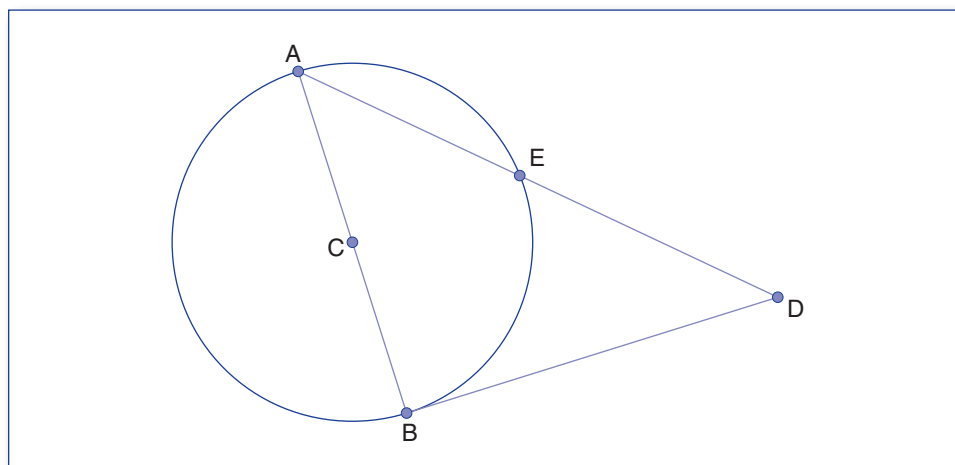
forms a right angle. This means that *ABD* is a right triangle, so the Pythagorean theorem can be applied to determine $\mathrm{m}\overline{AD}$. Proportional reasoning then becomes the key to solving the problem if one looks for a triangle similar to triangle *ABD* that contains $\overline{AE}$. This sort of proportional reasoning problem often appears on teacher certification exams (Educational Testing Service, 2009).

Determine m $\overline{AE}$ in Figure 10.1. Show all of your work and explain how you reached your conclusion.

STOP TO REFLECT

In addition to using proportional reasoning, Cuoco and colleagues (1996) noted that there are times when geometers use several languages at once and times when they use a single language for everything. It is often the case that multiple techniques can be used to solve a problem. Coordinate geometry and vectors are among the languages that can be brought to bear in solving a single geometric problem. While these languages often come to the aid of geometers, geometry itself assists many other branches of mathematics. Euclidean geometry, for example, provides language to talk about algebraic objects such as the coordinate (3, 4) (a point), $y = 3x + 1$ (a line), and $x + 2y + 3z = 1$ (a plane). Other branches of mathematics, such as number theory, can also be simplified by using the geometric concept of point. Number theorists call the points in the Cartesian plane with integer coordinates lattice points. Lattice points can be considered fundamental objects of study in number theory. Essentially, the relationship between the language of geometry and that of other branches of mathematics is reciprocal. Geometry often comes to the aid of other branches of mathematics, just as tools from other branches can be useful for geometric problems.

Another hallmark of geometric reasoning is the richness of the systems that geometers construct and work within. Euclid's geometric system was based on five postulates. Controversies surrounding the fifth postulate, often called the *parallel postulate*, provoked the creation of several alternative, non-Euclidean systems. Referring to Figure 10.2, the parallel postulate essentially states that if $\mathrm{m}\angle CAB + \mathrm{m}\angle DBA < 180°$, then $\overrightarrow{CA}$ and $\overrightarrow{DB}$ must eventually intersect on the side of $\overrightarrow{AB}$ where $\angle CAB$ and $\angle DBA$ are situated. Many efforts were made to prove that Euclid's parallel postulate was actually just a consequence of the previous four. One individual who attempted such a proof was Nikolai Lobachevsky. In the early 19th century, he came to believe that no such proof was possible, and his attention shifted to designing a valid geometric system based in part on an axiom directly contradicting the parallel postulate (Boyer & Merzbach, 1989). Upon the publication and widespread acceptance of his results, normative modes of geometric reasoning were permanently shaken. Euclidean geometry was no longer considered the sole arbiter of absolute truth, since Lobachevsky's geometry, and several others, proved to be logically consistent as well. Geometers henceforth did not restrict their investigations to the Euclidean plane, and they began to study new ideas such as spherical and hyperbolic geometry.

Figure 10.2 Illustrating Euclid's parallel postulate.
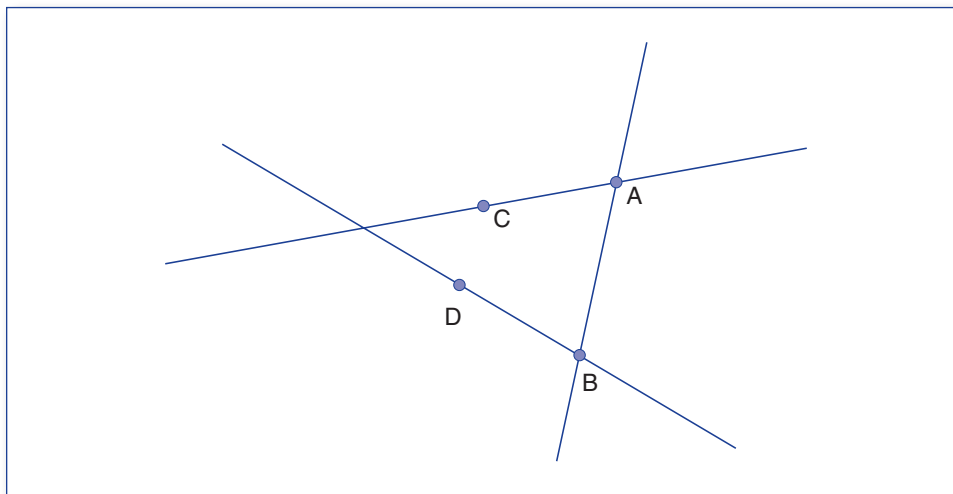
As geometers work within systems, they often explore characteristics of geometric objects that change as well as those that remain invariant. To illustrate the exploration of change, consider the three diagrams shown in Figure 10.3.

Figure 10.3 Changing the location of the intersection of two segments.
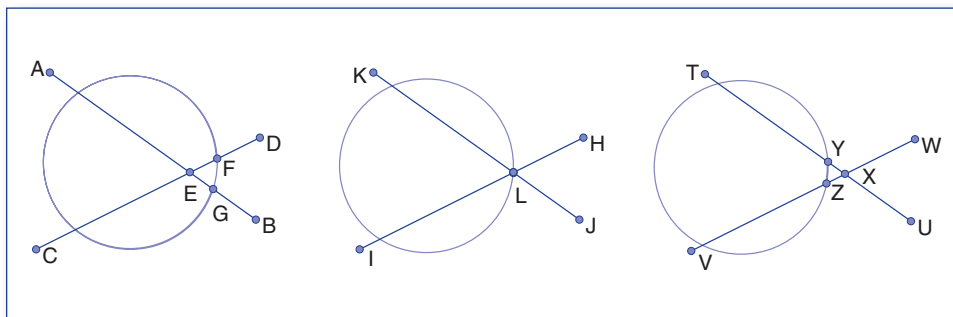
Geometers are interested in exploring such matters as how the arc lengths and angles in Figure 10.3 relate to one another. In the circle farthest to the left, the point of intersection between two chords lies in the interior of the circle. In the second diagram, the intersection point lies on the circle. Finally, in the third diagram, it lies on the exterior. Arc and angle relationships change as the point is moved. Along with change, invariance is interesting to geometers. Cuoco and colleagues (1996) noted that looking for invariants under geometric transformation is a particular point of interest. For example, identifying points that do not move under transformation can help one find the location of the center of a rotation.

Given the preceding examples of normative geometric thinking, it probably goes without saying that studying shapes is another fundamental geometric habit of mind. Three categories pertinent to the study of shape are classification, analysis, and representation (Cuoco et al., 1996). Shapes can be classified into categories based on attributes such as congruence, similarity, symmetry, self-similarity, and topology (Senechal, 1990). Analysis in geometry regularly includes looking for lines of symmetry, using lattices, and dissecting shapes (Senechal, 1990). Representations can come in the form of physical models of geometric objects, maps, shadows and lenses, drawings, and computer graphics (Senechal, 1990). These forms of representation support visual thinking needed for advancements in mathematics and science.

> **Implementing the Common Core**
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> See Homework Task 1 to use dynamic geometry software to explore the diagrams shown in Figure 10.3, which lead students to "understand and apply theorems about circles" and "find arc lengths" (Content Standard G-C).

Students' Geometric Thinking: van Hiele Levels

It is important for teachers to know that acquisition of normative geometric habits of mind is generally not a rapid process. In their doctoral dissertation work, Dina and Pierre van Hiele identified several levels of development through which students tend to pass in learning geometry (van Hiele, 1986). These are commonly referred to as the **van Hiele levels**. The van Hiele levels have been used extensively to guide investigations of students' geometric thinking. Researchers have invested a great deal of time in testing the levels against empirical classroom data. Though some have questioned the descriptive power of the levels and have proposed refinements to them, the van Hiele levels remain one of the most influential lenses for studying the development of students' geometric thinking.

Battista (2009) provided a summary of current thought on the van Hiele model and the characteristics of each level. Level 1 is referred to as **visual-holistic reasoning**. At this level, students can name shapes when they are shown to them. However, the names are based on the general appearance of the shapes rather than on careful analysis of their properties. A student reasoning at level 1, for example, may see rectangles and squares as completely different kinds of shapes simply because they perceive rectangles to be "longer" than squares. At this point, students characterize shapes by general appearance rather than by carefully comparing components such as sides and angles. Many level 1 students are also affected by the orientation of a shape. For instance, if a square is rotated from its conventional position, they may consider it to be a "diamond" rather than a square (see Figure 10.4).

Battista (2009) called van Hiele level 2 **descriptive-analytic reasoning**. At this level, students begin to differentiate among shapes by analyzing their component parts. Hence, rather than considering an object to be a rectangle because it is long and skinny, students begin to focus on properties such as angles at intersections of segments and how segments are oriented relative to one another. Students' descriptions of these properties may consist of informal language until the formal terms *point*, *segment*, and *angle* are learned in school. Although level 2 students begin to describe shapes in terms of their component parts, they do not make connections

Figure 10.4 The same square in two different orientations.
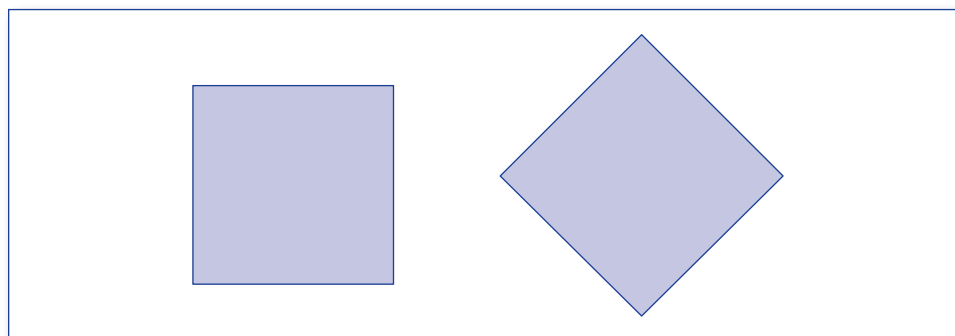
among the descriptions they give. For example, they may give definitions for *square* and *rectangle* that capture many of the pertinent characteristics of each shape, yet not understand how a square can be considered a special type of rectangle.

Students do begin to see relationships among definitions for geometric shapes at van Hiele level 3, **relational-inferential reasoning** (Battista, 2009). Inferences about the characteristics of shapes are generally made from observing many examples. For instance, after constructing and measuring many parallelograms, students may conclude that their opposite sides are always congruent because the property holds in all of the examples they consider. Constructing definitions for shapes by drawing on multiple examples prepares students to reason hierarchically about the definitions. Since squares and rectangles can both be described as quadrilaterals with four right angles, for instance, students can begin to see the logic of categorizing a square as a special type of rectangle. Even those who see this sort of logic, however, sometimes initially resist imposing a hierarchy on definitions.

Attainment of van Hiele level 4, **formal deductive proof** (Battista, 2009), is the goal of most high school geometry courses. At level 4, students understand the importance of undefined terms, definitions, axioms, and theorems in deductive reasoning. They can construct proofs by drawing on given information and using previous results to build a deductive argument. Common tasks at the high school level that require level 4 thinking include proving that two triangles in a diagram are similar or congruent by using theorems such as side-angle-side and angle-side-angle. High school geometry textbooks are often replete with such exercises.

Battista (2009) called the final van Hiele level **rigor**. At this level, students are able to reason about alternative axiomatic systems. They can understand that more than one logically consistent system of geometry exists. Although the study of non-Euclidean geometries is usually left to university-level mathematics courses, some mathematics educators advocate them as enrichments to conventional high school courses. House (2005), for example, provided ideas for teaching **taxicab geometry** at the secondary school level. Taxicab geometry redefines the conventional concept of distance in plane geometry. Superimposing a grid on the plane helps illustrate the difference in definitions (see Figure 10.5).

The left side of Figure 10.5 shows how distance is measured in conventional plane geometry. The distance between (−4, 3) and (5, −1) is measured "as the crow flies" and can be determined using the distance formula ordinarily taught in algebra.

Figure 10.5   Two different ways to measure the distance between (–4, 3) and (5, –1).
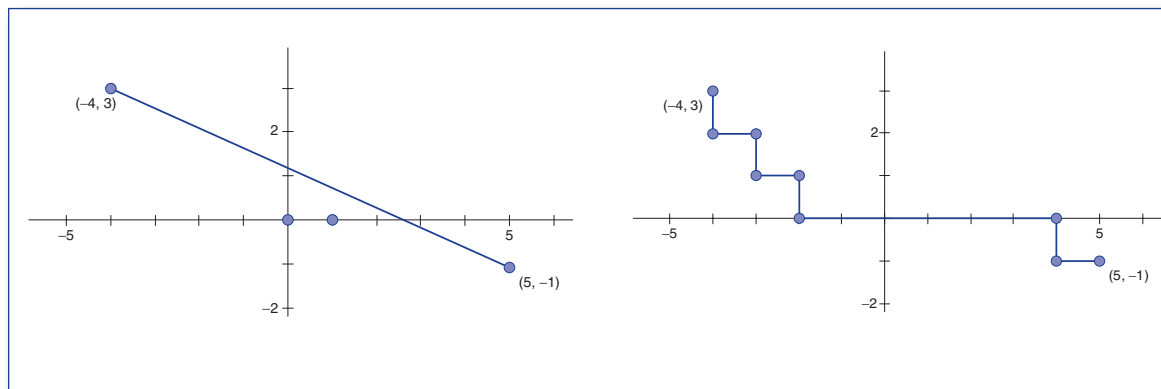
The right side of Figure 10.5 shows the measurement of distance in taxicab geometry. The coordinate grid can be thought of in terms of city streets, and distance is measured by considering the shortest path a taxi could drive in commuting between two points on the grid. Counting the number of unit grid lengths traced out by the shortest path gives the distance. Redefining the concept of distance in this manner can be used as a springboard for a task such as asking students to determine the conditions under which Euclidean and taxicab distances are the same and when they are different. Students may also be asked to describe what a circle would look like in taxicab geometry, given that a circle is defined as the set of all points equidistant from a given point (House, 2005).

Sketch several examples of taxicab geometry circles. Explain how taxicab geometry circles are similar to and different from Euclidean circles. Also write a distance formula that can be applied to find the distance between any two points in taxicab geometry, and compare and contrast this formula with the one for finding the Euclidean distance between two points in the Cartesian plane.

STOP TO REFLECT

Some of those who have challenged the descriptive power of the van Hiele model question whether students progress through the levels in a linear fashion. Gutiérrez, Jaime, and Fortuny (1991) argued that it is more accurate to speak of students' thinking in terms of degree of acquisition of van Hiele levels rather than as a progression of discrete jumps from one level to the next. Their data suggested that students may function at several different levels simultaneously. Despite this potential limitation, Gutiérrez (1992) found van Hiele levels to be useful for characterizing students' reasoning with three-dimensional objects. In addition, Jaime and Gutiérrez (1995) used van Hiele levels to describe students' reasoning about geometric transformations.

Perhaps the most vivid insights yielded by research on students' acquisition of van Hiele levels are the characterizations of students' thinking after completion of high school geometry courses. Recall that the goal of most high school geometry courses is to help students attain van Hiele level 4, deductive reasoning. In a study of 2,700 students from five different states, Usiskin (1982) found that most students did not progress beyond the first two van Hiele levels, even after completing high school

geometry. Senk (1985) underscored Usiskin's results by finding that only 30% of students from proof-oriented geometry courses attained 75% mastery in proving. Even those who demonstrated mastery often did not know the purpose of constructing proofs. Without such understanding, proof becomes a meaningless, mechanistic ritual to perform rather than a means of building knowledge in mathematics. Clearly, conventional high school geometry courses have fallen well short of the goal of helping all students attain van Hiele level 4.

The preceding discussion of the van Hiele levels suggests aspects of students' geometric thinking that need teachers' attention. Students' reasoning about fundamental geometric objects and their definitions is emphasized in the first three van Hiele levels. Students' ability to construct and understand proofs compose the core of the fourth level. Reasoning about geometric measurement is inherent in understanding both shapes and proof. A particular form of geometric measurement emphasized in school curricula is trigonometry. Some school geometry curricula are also beginning to emphasize more contemporary topics such as transformations, tessellations, chaos, and fractals. Therefore, students' patterns of thinking in regard to all of the preceding curricular areas will be explored next.

> **Implementing the Common Core**
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> See Clinical Task 2 to administer a comprehensive assessment of students' ability to construct geometric proofs (Content Standards G-CO, G-SRT, and G-GPE) and use the assessment data to drive instructional decisions.

Understanding Fundamental Shapes and Their Definitions

Students build their own personal definitions for shapes from examples they see in school and in everyday life. Through their experiences, students build **prototypes** that can become quite strong and influential in their thinking. Prototypes can be described as mental images that exemplify categories (Lakoff, 1987). Those that capture many of the relevant aspects of a category can be helpful in learning mathematics, and those that are more limited can be detrimental (Presmeg, 1992). Unfortunately, students often build limited prototypes for geometric objects, such as (1) an altitude always lies inside a triangle, (2) diagonals always lie inside a polygon, (3) right triangles have their right angles oriented toward the bottom in a diagram, and (4) the base of an isosceles triangle is positioned at the bottom of a diagram (Hershkowitz, Bruckheimer, & Vinner, 1987). In regard to numbers 3 and 4, individuals often find it more difficult to recognize isosceles and right triangles when presented with nonprototypical images. The influence that prototypes exert on individuals' thinking can make it difficult for them to understand and accept formal definitions that conflict with their entrenched prototypes. When possible, teachers should select unusual examples to share with students to challenge and uproot limiting, entrenched prototypes.

> **Implementing the Common Core**
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> See Clinical Tasks 3 and 5 to explore students' attention to precision (Standard for Mathematical Practice 6) when writing geometric definitions.

Another issue to deal with in teaching formal definitions for geometric objects is that there exist both **hierarchical** and **partitional definitions** (de Villiers, 1994). Hierarchical definitions, as the term suggests, establish a system in which a hierarchy of concepts can be formed. In the case of quadrilaterals, many textbooks adopt a hierarchical classification

scheme in which a square is a special type of rectangle, a rectangle is a special type of parallelogram, and a parallelogram is a special type of quadrilateral. A primary reason for adopting hierarchical definition schemes is that they simplify the process of deductive proof. For instance, under the hierarchical system suggested above, if one can prove that a property is true for all parallelograms, then the property automatically applies to all squares as well. Unfortunately, students often resist adopting hierarchical definition schemes because of entrenched prototypes that lead them to favor partitional ones. Partitional definitions, as the name suggests, partition concepts into separate, mutually exclusive bins. Many students have prototypical images of squares and rectangles that suggest one is not a subset of the other. This leads them to favor distinct definitions for the two concepts rather than overlapping ones.

IDEA FOR DIFFERENTIATING INSTRUCTION   Discussing Prototypes

Encouraging students to discuss the their personal prototypes for geometric objects is essential to helping them develop richer prototypes and accept more efficient hierarchical definitions. De Villiers, Govender, and Patterson (2009) suggested that it is inefficient for teachers to simply present definitions to students and expect them to commit them to memory. Instead, students should be encouraged to trace out some of the reasoning that went into the formation of the definition. One way to start engaging students in this reasoning process is to ask them to draw as many examples of a shape as possible. As students do this, teachers can gauge the range of personal prototypes students hold for the shape. Individual prototypes can then be shared and discussed publicly. In one instance (Groth, 2006b), this process helped a class negotiate a shared definition for *trapezoid*. Two definitions emerged as students shared drawings of their personal prototypes: (1) a quadrilateral with at least one pair of parallel sides and (2) a quadrilateral with exactly one pair of parallel sides. The class discussed the consequences of each definition in terms of which types of shapes would be considered trapezoids and which would not. Such considerations led some to favor one definition over the other. In formulating and debating the two definitions, the class was able to explore some of the considerations that go into creating geometric definitions, rather than just seeing the definitions in their finished form. Students at many different levels of understanding were able to participate in the process because gaining access to the task required reflection on personal prototypes rather than complete knowledge of formal definitions.

TECHNOLOGY CONNECTION

Dynamic Geometry Software, Drawings, and Constructions

Dynamic geometry software environments (DGEs) can also be used to help students form more powerful prototypes for concepts and understand formal definitions. DGEs such as Geometer's Sketchpad, Cabri, and GeoGebra can be used to prompt students to consider the fundamental characteristics of objects they are asked to construct. Consider the task of constructing a square in Geometer's Sketchpad. When asked to do

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so, many students produce a drawing of a square rather than a construction (Hollebrands & Smith, 2009). That is, they often use the segment tool to construct four segments that appear to be of equal length and perpendicular at the appropriate intersections. A **DGE drawing** ceases to be a square when dragged, but a **DGE construction** remains a square when dragged. See Figure 10.6 for

an illustration of how a drawing loses its "squareness" when dragged, and see Figure 10.7 for a construction of a square that remains a square when dragged. In general, objects that are drawn lose their properties when dragged, but those that are constructed maintain them. Prompting students to make constructions rather than drawings can help them carefully examine the properties of the given shape.
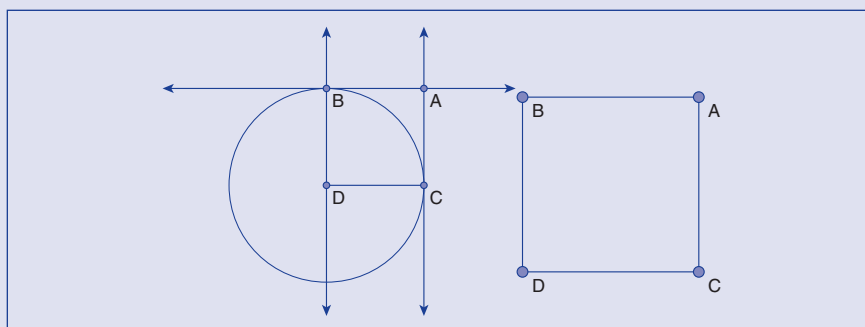
Figure 10.6   A DGE drawing of a square before and after dragging vertex C.
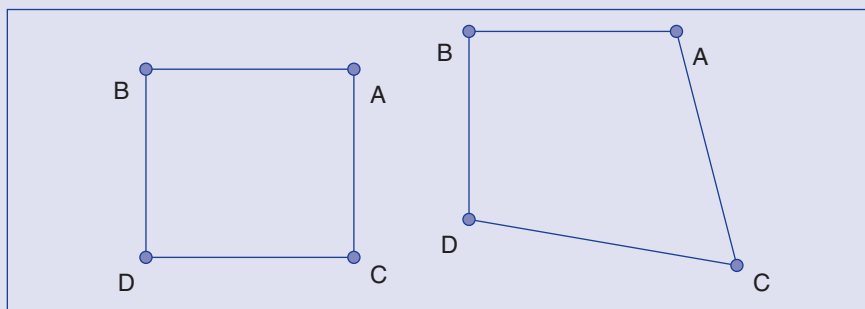
Figure 10.7   A DGE construction of a square before and after dragging vertex C.

The drawing in Figure 10.6 was produced simply by using the segment tool to construct four connected segments and arranging them into a square-looking object. When producing such a drawing, students may go so far as to use the angle and side measurement capabilities of the DGE to ensure that they have four right angles and four congruent sides. However, as soon as one of the sides or vertices is dragged, the drawing loses its squareness. This problem can be overcome by producing a construction as shown in Figure 10.7. To produce the square figure

shown, the segment tool was used to produce *DC*. Then the perpendicular line command was used to construct *BD*. The intersection point of circle D was then constructed with the appropriate software command, and another perpendicular line was constructed through point B. One last perpendicular line was constructed through point C, and its intersection with *BA* was constructed to form the fourth vertex of the square. Since the object of interest was the square, all extraneous por-

Implementing the Common Core

See Clinical Task 4 to assess a student's ability to construct a square with dynamic geometry software (Content Standard G-CO.12).

tions of the construction were taken away by using the appropriate Hide Objects command. Dragging sides and vertices in the construction produces new squares rather than destroying its squareness.

K. Jones (2000) found that asking students to produce constructions, rather than drawings, can help improve their understanding of formal definitions and relationships among geometric objects. He began instruction by asking students to produce objects in a DGE that could not be "messed up" by dragging. This request prompted students to move beyond simply producing drawings of the objects. In one case, students were to construct a rhombus and explain why it was a rhombus. In another case, they were asked to produce a rectangle that could be dragged to make a square, and then to explain why all squares can be considered rectangles. Later on, they were asked to construct a kite that could be dragged to produce a rhombus and explain why rhombi can be considered kites. By the point in the instructional sequence when the kite task was given, students had become increasingly formal in their use of mathematical statements. Instead of using informal language or the language of the software in their explanations, they had transitioned to using formal geometric language. This transition was demonstrated by the increasing sophistication of their statements about squares and rectangles. Initially, a square was considered to be a type of rectangle simply because it "looked like one." Using the DGE led students to refine their justification by stating that a rectangle can be "dragged into a square." Finally, at the end of the unit, students discussed the fact that both are "quadrilaterals with four right angles." Producing constructions of objects within the DGE led to this gradual adoption of normative geometric language and modes of thinking.

Implementing the Common Core

See Homework Task 3 to try your hand at making geometric constructions with dynamic geometry software (Content Standard G-CO.12).

Follow-up questions:

1. What other types of quadrilaterals could be constructed in a DGE? Explain how you would construct at least one other quadrilateral.

2. Why is it important for teachers to know the distinction between "drawings" and "constructions" in DGEs?

TECHNOLOGY CONNECTION

Analyzing Premade Dynamic Constructions

Another DGE-based strategy that can facilitate understanding of definitions of shapes and relationships among them is having students work with premade constructions rather than producing their own. The Shape Makers environment (Battista, 2003) supports such an approach. Shape Makers comes packaged

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with premade constructions for shapes such as squares, trapezoids, parallelograms, and rectangles. Yu, Barrett, and Presmeg (2009) described two types of tasks they asked students to do when using Shape Makers. In one type of task, students experimented with different shape makers to produce given pictures made of shapes (e.g., a person made of squares, rectangles, circles, and other shapes). Carrying out these tasks prompted students to examine the range of capabilities for each shape maker. In another type of task, students experimented with shape makers to see which ones would produce different shapes (a square, a rhombus, a trapezoid, etc.). As they worked through the tasks, students were encouraged to write down their thoughts and conjectures and share them with the instructor and with each other. Exploring the capabilities of each shape maker and making their thinking processes explicit helped students refine personal prototypes for the geometric shapes under consideration.

Follow-up questions:

1. In what types of situations would you want your students to work with premade constructions? In which situations would you want them to construct their own?

2. What kinds of conjectures about relationships between shapes would you expect students to make when interacting with Shape Makers? Which important geometry concepts could they learn in the process?

UNDERSTANDING AND CONSTRUCTING GEOMETRIC PROOFS

Developing normative, formal definitions for shapes lays a foundation for understanding geometric proofs. Such a developmental progression is suggested by the van Hiele levels, since level 4 thinking involves the capability to deal with proof. Unfortunately, as noted earlier, far too few students are successful in understanding proof upon completing their high school geometry courses (Senk, 1985; Usiskin, 1982). The following discussion describes some of the reasons for the widespread failure to understand proof and some steps that can be taken to remedy the situation.

The Intellectual Need for Deductive Proof

A major cause of the difficulty with proof in schools appears to be that students often feel no intellectual need to reason deductively (Hershkowitz et al., 2002). **Deductive reasoning** involves incorporating accepted statements such as theorems, postulates, and definitions into a logical argument. For example, if one wishes to prove that all triangles contain 180 degrees, a deductive argument that involves theorems about alternate interior angles could be used as a key aspect. Simply measuring the angles in several different triangles would not be sufficient, since it is not possible to measure every triangle that could possibly be constructed. However, students are often satisfied of the truth of a conjecture after seeing only a few specific cases where it holds up. Most of the geometry students Koedinger (1998) interviewed were satisfied that the diagonals of kites are perpendicular after seeing a few specific examples. Students with this tendency have been called **naive empiricists** (Balacheff, 1988) because they rely entirely on the empirical evidence produced by a finite number of cases.

Naive empiricism, though a prevalent cause of difficult with proof, is not the only cause. Some students do not even go so far as to gather empirical evidence to support a conjecture. Instead, they exhibit external proof schemes, believing that truth is established by appealing to outside authority (Harel & Sowder, 1998). In addition, even those who know the mechanics of deductive proof may not see an intellectual need for it. Some of the students interviewed by Koedinger (1998) were actually able to write a deductive proof showing that diagonals of a kite are perpendicular. However, these students usually only did so when the interviewer used the prompt "Do a proof like you do in school." Furthermore, some students who can follow a deductive proof may not be convinced that it covers all cases. Chazan (1993) found that some students wanted additional empirical examples to support a deductive proof after it had been written. These results support the idea that lack of intellectual need, perhaps even more than lack of knowledge of the mechanics of proof, is a key roadblock.
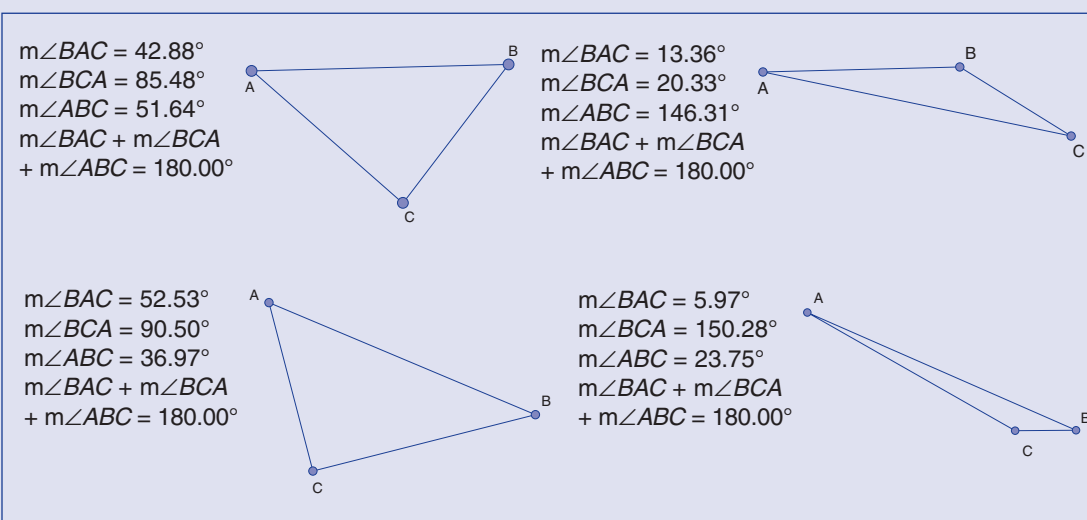
> **Implementing the Common Core**
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> See Clinical Task 6 to probe a student's intellectual need to prove a theorem about the sum of the measures of the interior angles of a triangle (Content Standard G-CO.10).

TECHNOLOGY CONNECTION

Using DGEs for Proof Activities

DGEs present both challenges and opportunities when it comes to helping students see the need for deductive proof. DGEs allow users to produce many examples very quickly. For instance, a student may quickly become convinced that the interior angles of all triangles sum to 180 degrees while dragging vertices and observing that the sum remains constant (Figure 10.8).

Figure 10.8 Empirical examples of triangle measures produced in a DGE.

$m\angle BAC = 42.88°$
$m\angle BCA = 85.48°$
$m\angle ABC = 51.64°$
$m\angle BAC + m\angle BCA$
$+ m\angle ABC = 180.00°$

$m\angle BAC = 13.36°$
$m\angle BCA = 20.33°$
$m\angle ABC = 146.31°$
$m\angle BAC + m\angle BCA$
$+ m\angle ABC = 180.00°$

$m\angle BAC = 52.53°$
$m\angle BCA = 90.50°$
$m\angle ABC = 36.97°$
$m\angle BAC + m\angle BCA$
$+ m\angle ABC = 180.00°$

$m\angle BAC = 5.97°$
$m\angle BCA = 150.28°$
$m\angle ABC = 23.75°$
$m\angle BAC + m\angle BCA$
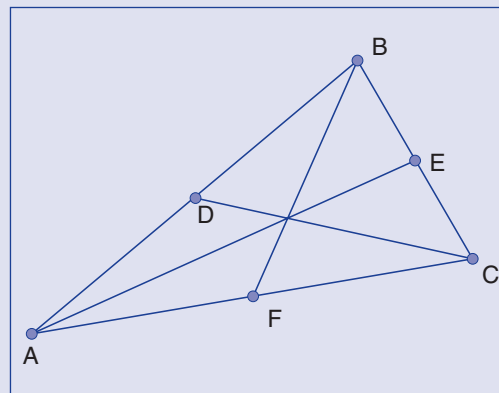$+ m\angle ABC = 180.00°$

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(Continued)

Since empirical examples are so easy to come by in a DGE, the potential exists for students to see even less need for deductive proof (Mariotti, 2001). However, it is also true that mathematicians often examine many empirical examples in order to become convinced that a proposition is worth attempting to prove (de Villiers, 1998). Although students, particularly naive empiricists, may become too confident in conclusions gained from analyzing empirical examples, analyzing examples intelligently is part of normative geometric thinking. Hence, it does not seem reasonable to prohibit students from using DGEs for proof-oriented activities. Instead of asking *if* students should be allowed to use DGEs, a more productive question is *how* teachers can help students use them appropriately.

One way to help students engage in productive work in a DGE is to carefully choose the questions they address with the technology. De Villiers (1998) recommended having students make conjectures about patterns they see when exploring within a DGE, and then asking them to explain why the conjectures are true. In one activity, for example, students were asked to construct a triangle and the midpoints on each of its sides. They were then to connect the midpoints to their opposite vertices to form the medians of the triangle (Figure 10.9) and state a conjecture about the medians. Students tested their conjectures by dragging the triangle and observing what happened to the medians as the triangle become obtuse, scalene, and right. After drawing tentative conclusions about the situation, students shared with one another. They were then asked to explain why the conjecture was true by explaining it "in terms of other well-known geometric results" (p. 392). After constructing deductive explanations, students again shared them with one another to identify areas of agreement and disagreement as well as the explanations that seemed most satisfactory. Such an approach stands in contrast to conventional proof instruction, where students are generally given statements and then asked to prove them. In de Villier's activities, students had roles in formulating the conjectures to be proven, which helped create an intellectual need to explain why the conjectures were true.

Hadas, Hershkowitz, and Schwarz (2000) used a different approach to establishing an intellectual need for deductive proof. They set up a situation where conjectures students formed while examining empirical examples proved to be incorrect. In an introductory activity, students were asked to determine the sum of the interior angles in a polygon, and to notice that the sum changed with the number of sides. From their observations, they were to make a conjecture about the sum of the exterior angles. Most students believed that the sum of the exterior angles would change with the number of sides, just as the sum of the interior angles had changed. They were surprised, however, when they examined more examples and found that the sum of the exterior angles was constant, regardless of the number of sides on the polygons they constructed in

Figure 10.9   Medians of a triangle.

a DGE. This sparked students' curiosity, and many felt a need to explain why the exterior angle sum remained constant. The students did not believe that the DGE constructions provided an explanation of why this was the case, so they set about reasoning deductively to form satisfactory explanations. In this situation, the conflict between expected and obtained results was a powerful catalyst in moving students toward formal geometric proof.

Hadas and Hershkowitz (1998) also used the element of surprise to help prompt students toward forming deductive proofs. They asked students to make a conjecture about a diagram where one of the angles of a triangle was trisected with two segments (Figure 10.10). Most students believed that the side opposite the trisected angle would be split into three congruent parts. When students tried to support this conjecture by measuring empirical examples, they were not able to produce any examples to support the conjecture. Students drew on geometric properties they had learned in the past to begin to explain why the conjecture actually was not true. As in the Hadas, Hershkowitz, and Schwarz (2000) study, the elements of uncertainty and contradiction of intuitive beliefs provoked an intellectual need for deduction.

Since naive empiricism is deeply ingrained in the thinking of many students as they study geometry, teachers should not expect rapid mastery of deductive proof. Moving from empirical observations to deductive reasoning is a process that takes time. DGE use is likely to be most effective in moving students toward proof when adequate instructional time is invested (Marrades & Gutiérrez, 2000). Though it may seem inefficient at first to allow extensive amounts of time for examining examples, making conjectures, and explaining why the conjectures are true, such a sequence of activities is optimal for helping students begin to engage in normative geometric thinking patterns.

> **Implementing the Common Core**
>
> See Homework Task 4 to prove a theorem about the medians of triangles (Content Standard G-CO.10).

Figure 10.10   A triangle with one angle trisected by two segments.
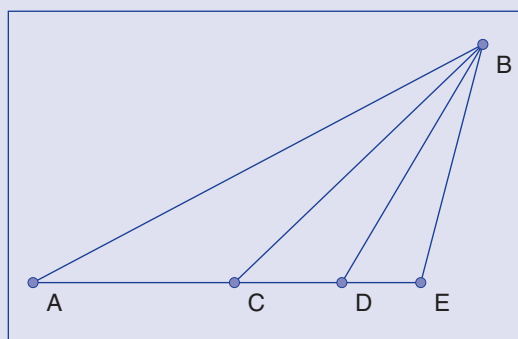
Follow-up questions:

1. Why is the sum of the exterior angles for any polygon 360°? Provide a deductive explanation.

2. Provide your own example of a counterintuitive geometric property that students could investigate in a DGE. How would students' DGE findings for your example conflict with their intuition?

Establishing a Classroom Culture of Proof

A DGE is just one possible element of a classroom environment that promotes a culture of proving. Martin, McCrone, Bower, and Dindyal (2005) studied a geometry classroom where students successfully made conjectures, provided justifications, and built chains of reasoning. To encourage these behaviors, the teacher posed open-ended tasks, placed responsibility for reasoning on the students, and analyzed their reasoning to determine when further coaching was necessary. In one classroom episode, the teacher began by giving students the open-ended task of writing down everything they knew about a pair of congruent pentagons. The task led one student to conjecture that the distances between nonadjacent corresponding vertices in each pentagon were equal. The teacher asked students to investigate the conjecture

through paper folding, affirmed the truth of the statement, and then asked students to explain why it was true. In response, students attempted to construct deductive explanations. Some of the attempts at deductive explanation were initially unsuccessful. The teacher introduced counterexamples highlighting the portions of the students' arguments that needed to be rethought. Students persisted in their attempts to construct deductive explanations because the teacher's feedback and interaction indicated that their attempts were valued, even if they were not initially correct.

Many teachers believe that providing a specific proof writing format is another way to support students' attempts to construct proofs. The two-column proof format, in which statements are written in one column and corresponding reasons in the other, is deeply ingrained in the culture of teaching geometric proof in the United States (Herbst, 2002). A typical two-column proof from high school geometry is shown in the top portion of Figure 10.11. In recent decades, the two-column format has been criticized. Schoenfeld (1988) remarked that constructing geometric proofs can become a ritualistic and mechanistic enterprise when the form of a proof is emphasized more than its substance. Moreover, mathematicians do not hold themselves to using the two-column format when constructing proofs. In an attempt to shift students' focus from the two-column format to the actual substance of proofs, *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Math-

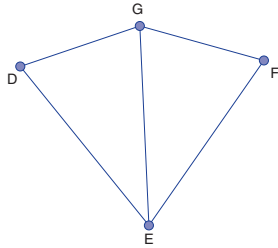### Implementing the Common Core

See Clinical Task 7 to inquire about a teacher's strategies for using different formats to help students construct proofs (Content Standards G-CO, G-SRT, and G-GPE).

ematics [NCTM], 1989) identified two-column proof as a topic that should receive less attention. Proof itself was still to be an important part of the curriculum, but NCTM recommended de-emphasizing the two-column format in an attempt to shift attention toward the quality of students' deductive reasoning and away from their ability to adhere to a specific format. Two other possible formats for writing proofs, flowcharts (McMurray, 1978) and paragraphs (Brandell, 1994), are shown in Figure 10.11.

Any given format for writing a proof has potential weaknesses and strengths. A potential weakness of any form is that students may begin to focus more on form than on substance, as noted earlier. The primary strength of paragraph proofs is that they closely resemble the types of proofs constructed by mathematicians. Therefore, students who can read and construct paragraph proofs may be in better position to succeed in college mathematics. Nonetheless, other formats may be useful for scaffolding students' thinking so that they can ultimately master paragraph proof. Even the two-column format, which has been somewhat demonized in the recent past, can serve a useful scaffolding role when used appropriately. Weiss, Herbst, and Chen (2009) noted that the two-column format can be a useful tool for outlining the general structure of a proof. In one classroom they observed, a student sketched a general structure for a proof using two columns, initially skipping some of the reasons in the second column but returning to fill them out later. When used in this way, the two-column format can help students organize their thinking. On the other hand, when teachers insist that two-column proofs be filled out in a linear fashion, with each step justified before another step may be written, students' thinking is constrained. In general, any proof format is valuable when used to help facilitate thinking rather than impede it.

Under any form of proof, care must be taken to ensure that students interpret the accompanying diagrams correctly. Battista (2007) described a variety of ways that students misinterpret the intended meanings of geometric diagrams. One common misinterpretation is to believe that a deductive proof only covers the specific diagram

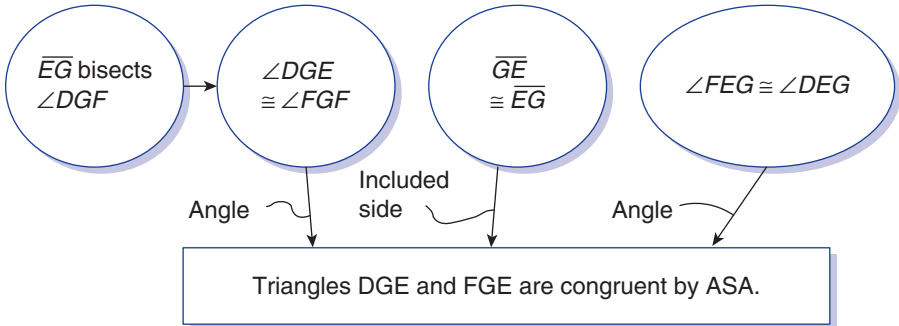**Figure 10.11**   The same proof written in three different formats.

Suppose we know that $\overline{EG}$ bisects ∠DGF and that ∠FEG ≅ ∠DEG. Prove that triangles DGE and FGE are congruent.

Two-column proof:

Statements	Reasons
1. $\overline{EG}$ bisects ∠DGF	1. Given
2. ∠DGE ≅ ∠FGE	2. The two angles formed by an angle bisector are congruent.
3. $\overline{GE}$ ≅ $\overline{EG}$	3. Reflexive property
4. ∠FEG ≅ ∠DEG	4. Given
5. Triangles DGE and FGE are congruent.	5. Angle-side-angle postulate

Flowchart proof:

$\overline{EG}$ bisects ∠DGF → ∠DGE ≅ ∠FGF

$\overline{GE}$ ≅ $\overline{EG}$

∠FEG ≅ ∠DEG

Angle   Included side   Angle

Triangles DGE and FGE are congruent by ASA.

Paragraph proof:

The two triangles can be shown congruent by using the angle-side-angle congruence postulate. First, note that ∠DGE ≅ ∠FGE because $\overline{EG}$ bisects ∠DGF, and the bisector of an angle splits it into two congruent angles. Next, we know that $\overline{GE}$ ≅ $\overline{EG}$ by the reflexive property. Finally, it was given that ∠FEG ≅ ∠DEG. This shows that two corresponding angles in each triangle and their included corresponding sides are congruent. Hence, triangles *DGE* and *FGE* are congruent by the angle-side-angle (ASA) congruence postulate. Q.E.D.

accompanying the proof. In reference to Figure 10.11, for example, some students may believe that if $\overline{EG}$ were lengthened, a new proof would be required, even though the essential structure of the situation would remain unchanged. Another common misinterpretation is to believe that features of a diagram can disprove a theorem established deductively. Suppose, for example, a teacher drew a diagram of a circle and a tangent line, and then drew a radius out to the tangent line that appeared to intersect it at an acute angle. Such a diagram will lead some students to believe that the intersection of a radius and a tangent line does not always form a right angle. To help students avoid misinterpretations like these, it is important to explicitly discuss the meanings of geometric diagrams. Students need to understand that diagrams simply serve as (sometimes imperfect) visual props for working toward a deductive proof.
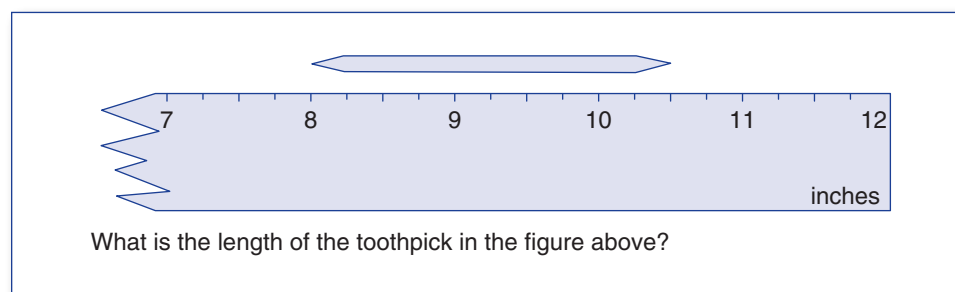
GEOMETRIC MEASUREMENT

Along with the study of shapes and proofs, measurement undergirds most geometry courses taught in middle and high schools. As in other areas of the mathematics curriculum, there are many student thinking patterns that should be taken into account in planning instruction. Students' thinking about measurement of length, area, volume, and angles is discussed below.

Measuring Length

An item from the National Assessment of Educational Progress (NAEP), commonly called the "broken ruler problem," provides a good starting point for discussing students' understanding of length measurement. The version of the problem given to students in 2003 is shown in Figure 10.12.

Figure 10.12   Broken ruler problem from 2003 National Assessment of Educational Progress.

What is the length of the toothpick in the figure above?

Source: http://nces.ed.gov/nationsreportcard/naepdata/.

Approximately 42% of eighth graders (Blume, Galindo, & Walcott, 2007) and 20% of high school seniors (Struchens, Martin, & Kenney, 2003) answer the broken ruler problem incorrectly. There are several ways students can go wrong. Some have difficulty reading fractional lengths. Even those who can do so may say that the toothpick is 10½ inches long because they are accustomed to reading the location of the endpoint on the right side of the ruler to determine a measurement. Those who do attempt to determine the distance between the two endpoints of the toothpick may do so incorrectly by counting each mark above a whole number as one unit of length. That is, since the whole numbers 8, 9, and 10 fall within the length of the toothpick, students may incorrectly conclude that the toothpick is 3½ inches long.

> **Implementing the Common Core**
>
> See Clinical Task 8 to investigate students' attention to precision (Standard for Mathematical Practice 6) when measuring length with a broken ruler.

Miscounting the units in the length of an object like the toothpick in the broken ruler problem suggests a fundamental misunderstanding of length measurement. Students need to understand that the distance between two consecutive whole numbers, rather than a mark above a number, represents a unit of length. One strategy for helping students understand this important characteristic of measurement involves setting aside formal units of measure and rulers. Van de Walle (2001) recommended that students measure length by iterating an informal unit along an object. For example, students might use their hand spans to measure the length of a table. Doing so reinforces the idea that one essentially lays the same unit end to end over and over again to measure length. When repeatedly iterating the informal unit becomes tedious, students can abbreviate the process by making their own measuring sticks by taping together several copies of the unit. This can help them understand how conventional rulers abbreviate the iteration process. Finally, when students obtain different measurements for the same table because their hands are different lengths, the concept of formal units can be introduced and appreciated for its ability to facilitate discussions about the length of an object.

Understanding how the iteration of units composes the foundation of measurement can also be useful when students attempt to estimate. Another problem that causes difficulty on the NAEP involves estimating the length of one object with another. On the 2003 NAEP, students were asked to estimate the length of the 882-foot-long cruise ship *Titanic.* They were to choose the most accurate estimate from among several options: 2 moving van lengths, 50 car lengths, 100 skateboard lengths, 500 school bus lengths, or 1,000 bicycle lengths. Only 39% of eighth graders answered correctly (Blume et al., 2007). The low rate of success on the item suggests a lack of opportunity to think about measuring an object through the iteration of units. Although such opportunities would ideally occur in the lower grades, middle and high school teachers may need to address gaps in students' understanding by providing opportunities for measuring via iteration of units in the later grades.

Another way to build students' understanding of length measurement involves using a **geoboard**. A geoboard is essentially a pegboard on which rubber bands can be strung to create geometric objects. Dot paper can be used in place of a physical geoboard. Some online applets also replicate physical geoboards (Figure 10.13). Ellis and Pagni (2008) described an instructional sequence for using the geoboard to help students understand lengths not represented by whole numbers. They asked students to determine the lengths of the sides of simple objects, such as squares and rectangles, formed on the geoboard. Students could do so by counting the number of units (rather than the number of pegs) along each side. They then asked students to determine the lengths of diagonals on the geoboard. This prompted students to use the Pythagorean theorem. If a length of 1 is assigned to the distance between neighboring pegs on the geoboard, then the shortest diagonal is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Once that length has been established, students can take on more complicated problems, such as determining how many segments of $\sqrt{10}$ can be found on the geoboard.

> ### Implementing the Common Core
>
> See Homework Task 5 to try your hand at a geoboard task that requires students to "understand and apply the Pythagorean Theorem" (Content Standard 8.G).

Figure 10.13 A geoboard from the National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vLibrary.html).
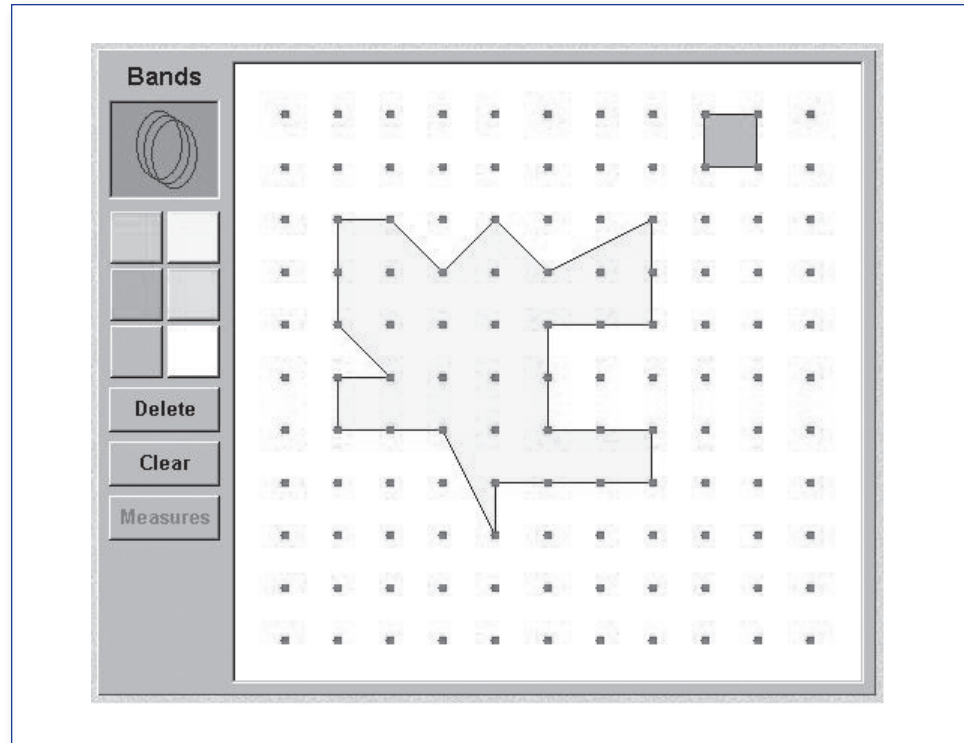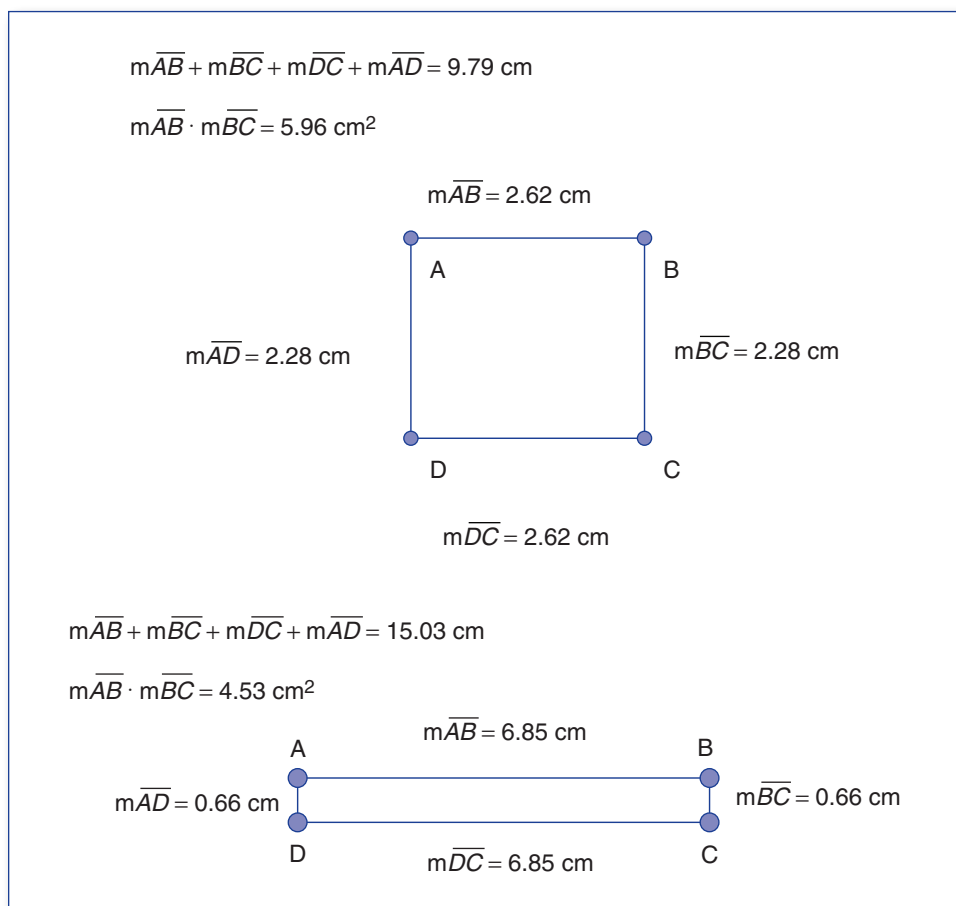

Measuring Area

Students' difficulties in measuring length feed difficulties in measuring area. NAEP data indicate that only about 24% of eighth graders can determine the surface area of a given rectangular prism (Blume et al., 2007). In addition, teachers are sometimes uncertain about relationships between perimeter and area of geometric objects. Ma (1999) asked teachers how they would respond to a student who claimed that the area of a closed figure always increases as its perimeter increases. Many of the teachers from the United States responded that they would ask the student to produce several examples to verify the claim. Few responded that they would guide students toward counterexamples that would actually refute the claim, like the one shown in Figure 10.14.

One of the prominent portions of the curriculum where measurement of length and area interact is the Pythagorean theorem. Although the Pythagorean theorem is often used simply to determine the length of a side of a right triangle when only the other two side lengths are provided, it also expresses an interesting relationship

> **Implementing the Common Core**
>
> See Clinical Task 9 to explore how well students "construct viable arguments" (Standard for Mathematical Practice 3) when discussing the relationship between area and perimeter of a shape.
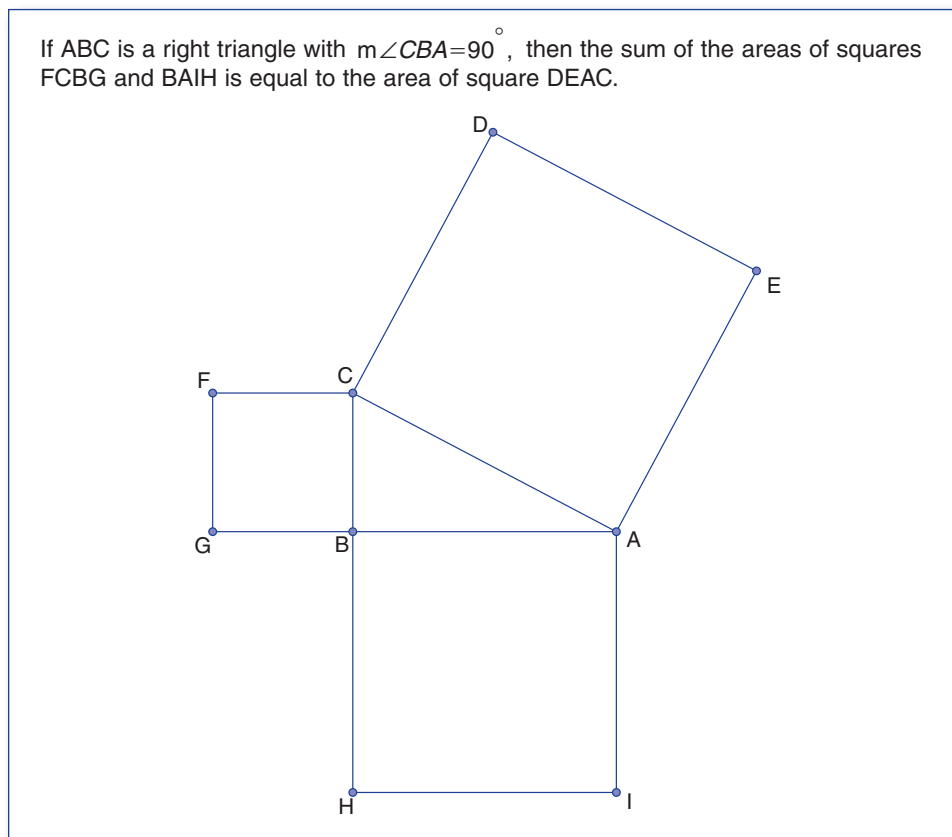
Figure 10.14   Decreasing area while increasing perimeter.

$m\overline{AB} + m\overline{BC} + m\overline{DC} + m\overline{AD} = 9.79$ cm

$m\overline{AB} \cdot m\overline{BC} = 5.96$ cm²

$m\overline{AB} = 2.62$ cm

A   B

$m\overline{AD} = 2.28$ cm $m\overline{BC} = 2.28$ cm

D   C

$m\overline{DC} = 2.62$ cm

$m\overline{AB} + m\overline{BC} + m\overline{DC} + m\overline{AD} = 15.03$ cm

$m\overline{AB} \cdot m\overline{BC} = 4.53$ cm²

$m\overline{AB} = 6.85$ cm

A   B

$m\overline{AD} = 0.66$ cm $m\overline{BC} = 0.66$ cm

D $m\overline{DC} = 6.85$ cm   C

among areas of squares. Essentially, it states that the sum of the areas of the squares constructed on the legs of a right triangle will equal the area of the square constructed on the hypotenuse (Figure 10.15). There are many ways to help students understand the Pythagorean theorem in terms of area. Yun and Flores (2008), for example, suggested using jelly beans as informal units of area measurement. They constructed several cardboard containers in the shape of the diagram shown in Figure 10.15. They then asked students to fill each square portion with a layer of jelly beans and describe the relationships among the areas of the squares. Yun and Flores recommended extending the activity by having students examine containers with circles and other shapes constructed on the sides of the right triangle to conjecture whether or not the relationship between the areas still holds. As in length measurement, work with informal units can be followed by work with formal units represented by unit squares.

Implementing the Common Core

See Homework Task 6 for an opportunity to "explain a proof of the Pythagorean Theorem" (Content Standard 8.G.6).

Figure 10.15   Expressing the Pythagorean theorem in terms of area.

If ABC is a right triangle with $m\angle CBA = 90°$, then the sum of the areas of squares FCBG and BAIH is equal to the area of square DEAC.

IDEA FOR DIFFERENTIATING INSTRUCTION   Measuring Nonrectangular Areas

Encouraging students to measure nonrectangular areas with unit squares sets the stage for diverse, rich thinking strategies to emerge. Hodgson, Simonsen, Lubek, and Anderson (2003) described an activity that required students to measure the area of the state of Montana. When presented with the task, students suggested superimposing a grid of unit squares on a map. They then proposed methods to obtain more accurate measurements, such as making the squares progressively smaller to minimize empty spaces within the grid. Utley and Wolfe (2004) suggested using the unit squares on geoboards as a means of measuring the areas of different shapes. Since geoboards have unit squares built into their structure, they can be used to visualize area for a variety of shapes, such as the trapezoid shown in Figure 10.16. One might form a rectangle around the

Implementing the Common Core

See Homework Task 7 to explore how a geoboard can be used for problems that require students to solve mathematical problems involving areas of unusual shapes (Content Standards 6.G and 7.G).

trapezoid and then cut out two triangles, or cut the shape itself into a rectangle and two triangles. Students may also devise alternative strategies for determining its area. It is important to note that knowing the formula for the area of a trapezoid is not necessary to perform the task. Determining the area is actually a good precursor to deriving a formula, as students can

be prompted to find shortcuts for calculating area after using more time-consuming visual methods repeatedly. The formula can then be understood and appreciated as an abbreviation of the visual methods rather than simply as a teacher-invented recipe for determining area. Students who discover a formula for the area of a trapezoid in advance of others can be encouraged to devise additional area formulas for more complex geometric shapes.

Figure 10.16   A geoboard representation of a trapezoid.
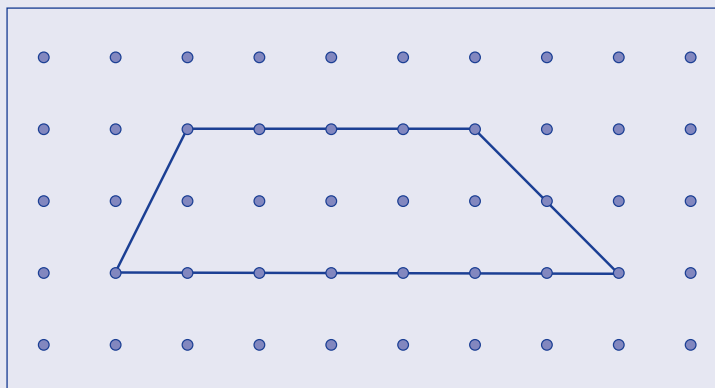
Describe at least four different ways to determine the area of the trapezoid shown in Figure 10.16. Explain your reasoning completely.

STOP TO REFLECT

Measuring Volume

As with area measurement, it is important for students to develop conceptual knowledge of volume before dealing with procedural formulas. Students who are very good at memorizing and using formulas to determine volume often have little understanding of what the formulas mean. NAEP results show that approximately 45% of 12th-grade students do not know that 48 cubic inches represents a measure of volume (Battista, 2007). Without knowledge of the fundamental unit that composes volume, students have little chance to develop conceptual understanding of volume measurement.

Implementing the Common Core

See Clinical Task 10 to investigate a student's ability to decompose a trapezoid into simpler shapes (Content Standard 6.G.1) to determine its area.

Battista (1999) described an instructional sequence for helping students develop conceptual understanding of volume. Instruction began by asking students to determine the number of cubes it would take to fill different boxes represented by two-dimensional drawings (see, for example, the diagrams in Figure 10.17). Students made conjectures, compared them with one another, and then tested the conjectures using actual cubes and boxes. Discrepancies between predicted and actual results prompted students to go back and revise their thinking. In some cases, students found their predictions to be incorrect because they double-counted cubes or omitted cubes in the middle of a box in forming their predictions. Some students began the task of counting the number of cubes in each box by using skip counting, but gradually moved to multiplication as a more efficient strategy. After working with

several boxes (i.e., rectangular prisms), students moved on to explore other solids, such as pyramids. At the end of the instructional sequence, students were able to successfully enumerate cubes in 3D arrays. Battista's instructional sequence stands in stark contrast to typical units that begin by introducing students to formulas for volume and then spend most of the allotted time having students repeatedly practice computation with those formulas.

Measuring Angles

Just as volume measurement can be difficult for students to understand conceptually, angle measurement often presents a significant cognitive hurdle. Some students believe that the lengths of the rays that make up an angle influence its measure (i.e., as the rays become longer, the angle measure increases even though the rays remain in the same orientation to one another; Struchens et al., 2003). Students also have a difficult time conceiving of angle measurement as an amount of turn (Mitchelmore & White, 2000). This particular student difficulty can be partially explained by the language teachers use to describe angles. Browning and Garza-Kling (2009) found that prospective teachers tend to describe angles as "corners" or "something you measure in degrees." Such descriptions of angles fail to emphasize their usefulness for measuring the amount of turn from one position to the next.

Given students' difficulty in conceiving of angle measurement as an amount of turn, geometry instruction should explicitly address this characteristic. Browning and Garza-Kling (2009) fostered understanding in this area by asking students to imagine that the degree, a standard unit of angle measurement, had not yet been invented. Students were to devise their own strategies for measuring angles. They settled on forming wedges and iterating them until they had completely measured out an angle. Since each group of students began with a different-size wedge, the importance of

Figure 10.17   A net and a representation of its three-dimensional structure when folded.
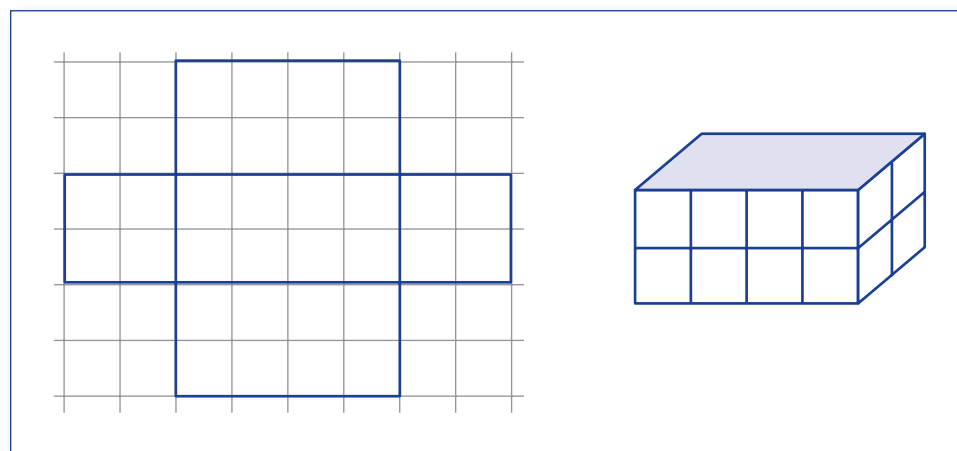
having a standard wedge to measure all angles became apparent so that each group would assign the same measure to a given angle. This led to understanding and appreciation of a degree as a standard "wedge size" used for angle measurement. Later on, Browning and Garza-Kling asked students to work with a graphing calculator applet showing angles being swept out by turning segments. Such experiences highlighted important concepts that compose the foundation of angle measurement.

> ### Implementing the Common Core
>
> See Homework Task 8 for an opportunity to design a manipulative to help students understand the concept of angle measurement (Content Standard 8.G).

TRIGONOMETRY

Many of the geometric concepts discussed so far, particularly the idea of angle, form the core of secondary school trigonometry. Despite the importance of the subject, research on the teaching and learning of trigonometry is in its infancy. It will be important for teachers to track developments in this field of research in the coming years as it continues to develop. Insights from existing research are discussed next.

In recent years, researchers have highlighted potential conceptual difficulties with common approaches to teaching trigonometry. P. W. Thompson (2008) claimed that the trigonometry of right angles and the trigonometry of periodic functions are often treated in isolation. Weber (2008) agreed, noting that the calculation of ratios in static triangles is often overemphasized in comparison to time spent building functional understanding. Thompson went on to note that trigonometry is often taught procedurally, with little emphasis on the conceptual underpinnings of angle measurement. Because of this, students may be able to transition between radians and degrees without understanding that the two are essentially just different units for measuring angles. In trigonometry, as in many other areas of the mathematics curriculum, fundamental problems with teaching and learning appear to be often rooted in an imbalance between emphases on procedural and conceptual knowledge.

Weber (2008) described a teaching sequence that departs from conventional approaches to trigonometry. He started instruction by asking students to work with a circle with a radius of 1 and centered on the point (0, 0) (i.e., a **unit circle**) on graph paper. On the unit circle, students drew angles and approximated the values of different trigonometric functions by measuring the coordinates of the intersection of the terminal segment and the unit circle. In Figure 10.18, for instance, students could estimate sine by estimating the y-coordinate of the intersection and cosine by estimating the x-coordinate. After associating sine with y-coordinates and cosine with x-coordinates, Weber had students estimate sine and cosine for several different examples. Along the way, students were asked to determine the exact values for sine and cosine of $0°, 90°, 180°, 270°$ and $360°$ without measuring. With this background, students were prepared to approach conceptual trigonometric tasks, such as determining whether or not it is possible to have a situation where $\sin(x) = 2$, deciding on the sign of $\cos(300°)$, and determining if $\sin(23°)$ is larger or smaller than $\sin(37°)$. Weber (2005) reported that this general approach helped students develop a deep understanding of trigonometric functions.

Figure 10.18 Unit circle diagram.
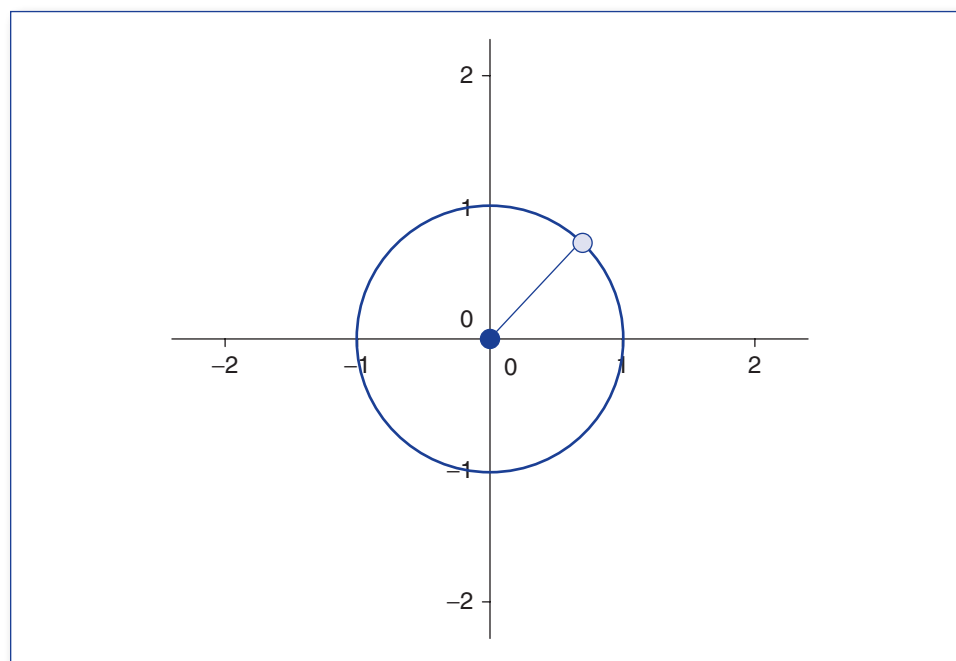
Cavey and Berenson (2005) described how teachers can improve their understanding of right triangle trigonometry by engaging in a modified version of Japanese lesson study. They traced the learning of one preservice teacher, Molly, during the process of collaborative planning and modification of lessons. Initially, when asked to teach a unit on right triangle trigonometry, all Molly could recall about the topic was the mnemon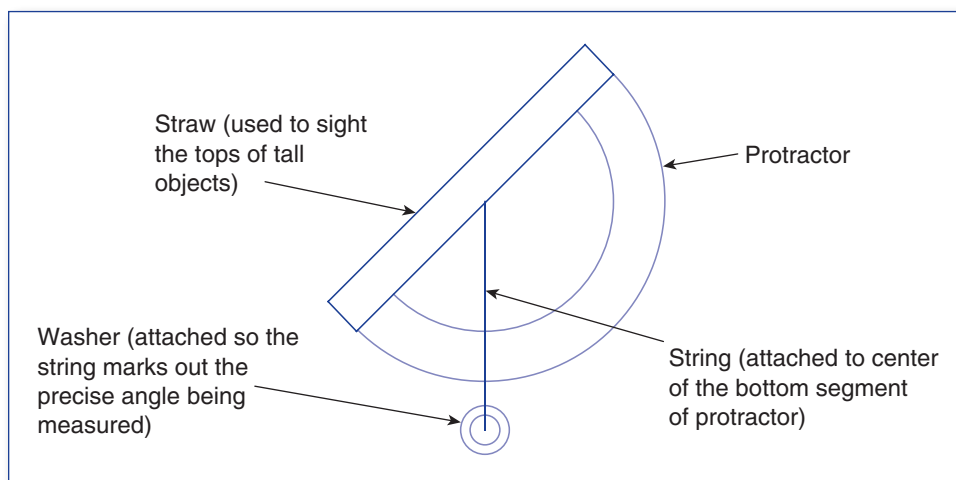ic "SOHCAHTOA." The acronym provided a means for calculating sine, cosine, and tangent, but little else (i.e., the acronym states that sine is "opposite [O] over hypotenuse [H]," cosine is "adjacent [A] over hypotenuse [H]," and tangent is "opposite [O] over adjacent [A]"). Through collaborative planning, Molly was able to take her teaching of trigonometry beyond this simple mnemonic. One idea she gained, for example, was using a **clinometer** (Figure 10.19), a tool consisting of a protractor, straw, washer, and string that can be used to measure angles in real-world situations involving right triangle trigonometry. In revising and extending her lessons, she also enriched her understanding of the mathematical concepts of ratio and similarity. Her improved content knowledge helped enhance the lessons she taught. Thinking about the mathematics within the context of her own practice proved to be a crucial element in Molly's development. Sharing her plans with others was a key mechanism in helping her identify and address gaps in her mathematical and pedagogical knowledge.

Implementing the Common Core

See Clinical Task 12 to assess students' ability to "solve problems involving right triangles" (Content Standard G-SRT) and "extend the domain of trigonometric functions using the unit circle" (Content Standard F-TF).

Implementing the Common Core

See Clinical Task 13 to assess students' ability to use trigonometric ratios to solve applied problems (Content Standard G-SRT.8) with a clinometer.

Figure 10.19   Components of a clinometer.

Straw (used to sight the tops of tall objects)

Protractor

Washer (attached so the string marks out the precise angle being measured)

String (attached to center of the bottom segment of protractor)

Describe how students could use a clinometer to measure the heights of very tall objects, such as the school building or trees surrounding it. Write at least three exercises you would ask students to do outside using clinometers. Include a diagram showing proper use of the clinometer to measure a tall object.

STOP TO REFLECT

CONTEMPORARY TOPICS IN GEOMETRY

As geometry continues to develop as a discipline, the school curriculum should respond accordingly. There has, of course, been much advancement in the discipline of geometry since the time of Euclid, yet the study of plane geometry and traditional measurement continue to dominate the curriculum. Contemporary topics that can be included in secondary curricula are transformation geometry, tessellations, fractals, and chaos (National Governor's Association for Best Practices & Council of Chief State School Officers, 2010; NCTM, 2000). As these ideas are relatively new to school curricula, we are just beginning to investigate optimal ways of teaching and learning them. Nonetheless, existing mathematics education research does provide some useful insights.

Transformation Geometry

Isometries can be defined as transformations in the plane that preserve the distance between points (Jaime & Gutiérrez, 1995). Isometries typically included in the high school curriculum include translations, rotations, and reflections. Isometries can be produced by a variety of methods, including paper folding, using dynamic geometry software,

Implementing the Common Core

See Homework Task 9 to explore a variety of transformation geometry problems (Content Standards 8.G, G-CO, and G-SRT) that can be approached using a Mira manipulative.

using online applets such as those available on the National Library of Virtual Manipulatives, and employing a Mira tool. Figure 10.20 shows isometries involving a hexagon.

Figure 10.20 Isometries involving a hexagon.

Horizontal translation   120° Rotation about point P   Reflection across segment *j*
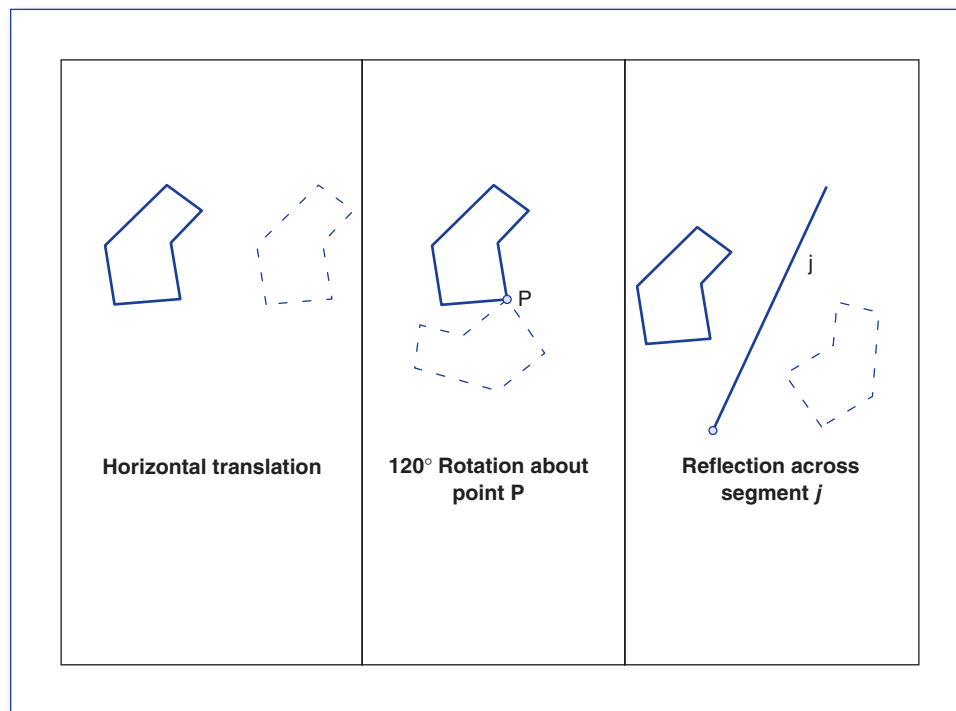
Jaime and Gutiérrez (1995) suggested using the van Hiele levels as a structure for determining goals for the study of plane isometries. They identified van Hiele level 3 as a suitable target for secondary school activities involving isometries, since most courses at this level aim to help students construct chains of deductive reasoning. Aiming for van Hiele level 3 means going beyond having students produce transformations. Jaime and Gutiérrez recommended tasks such as asking students to explain why the product of rotations is equivalent to a translation when the sum of the rotation angles is a multiple of 360°. Another recommended task was to explain why the product of two rotations is either a rotation or translation. Writing explanations for why transformations behave as they do can help students build the deductive reasoning skills characteristic of the higher van Hiele levels.

Tessellations

Students who have studied some transformation geometry can appreciate **tessellations**, which are tilings of a plane that do not contain any gaps or overlaps. Three regular polygons will tile the plane in this manner: triangles, squares, and hexagons. These three tile the plane because the measures of their interior angles are divisors of 360°. Figure 10.21 shows tilings done using an applet on the National Library of

Virtual Manipulatives website. Using applets, software programs, or just paper and scissors, students can make and test conjectures about the kinds of shapes that will tile the plane.

Figure 10.21   Tessellations using regular polygons.
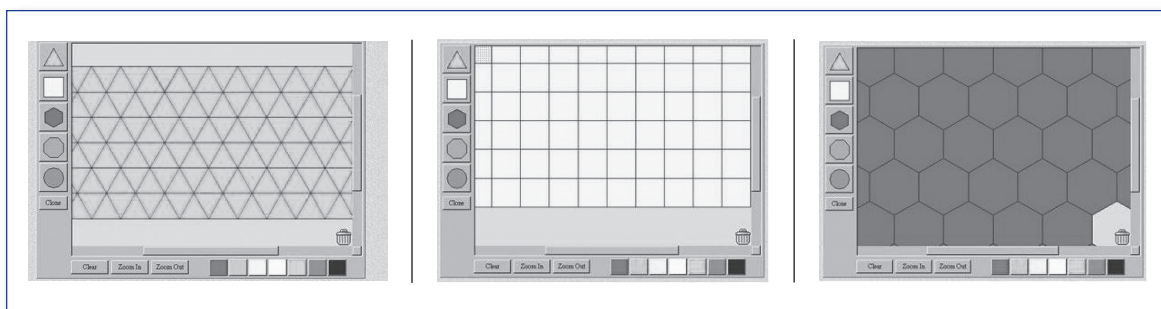
Perhaps the best-known tessellations occur in the artwork of M. C. Escher. Examples of how Escher incorporated tessellations in his paintings can be seen on www .mcescher.com.

Escher went beyond merely tiling the plane with shapes. He performed transformations on the shapes to form unique figures and then used them to tile the plane. Shockey and Snyder (2007) described an approach to helping students produce Escher-like tessellations. They asked students to take a square, cut a design along one edge of it (corner to corner), and then translate the design to the opposite side of the square (Figure 10.22). When the design is used to tile the plane, an Escher-like picture is formed. One could perform the same procedure with the other pair of opposite sides of the square to produce a different portrait. Students can also experiment to find other shapes and transformations that tile the plane.
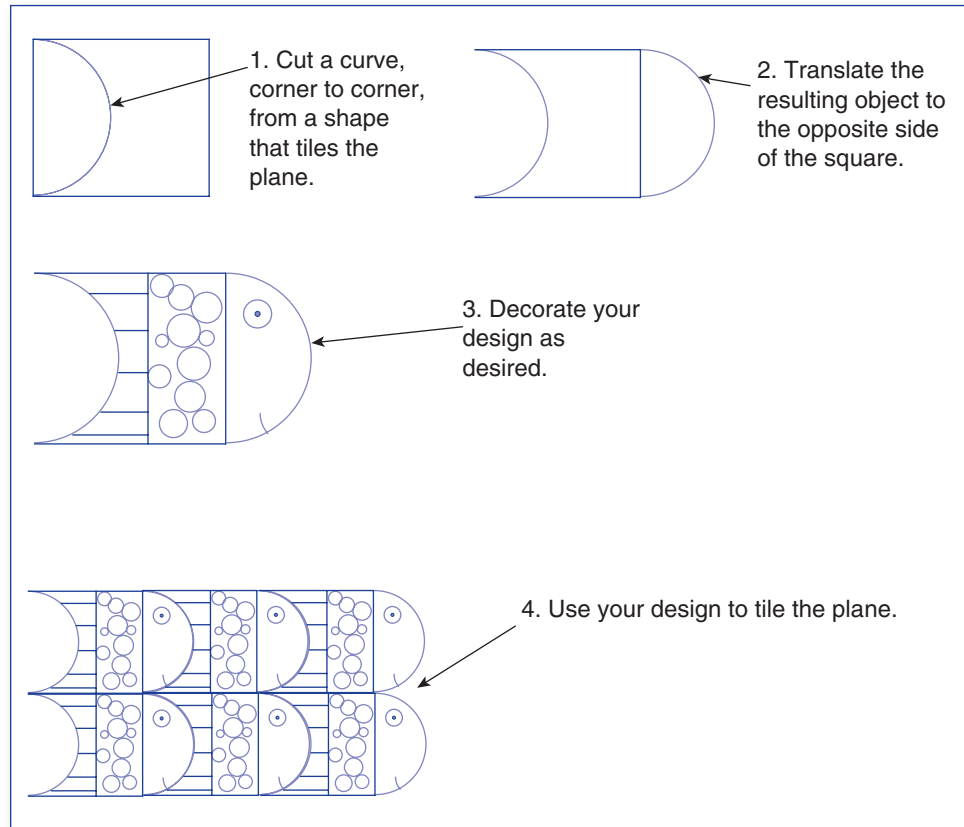
Fractals and Chaos

Concepts from fractal geometry have begun to make their way into middle and high school curricula. **Fractals** can be described in the following terms:
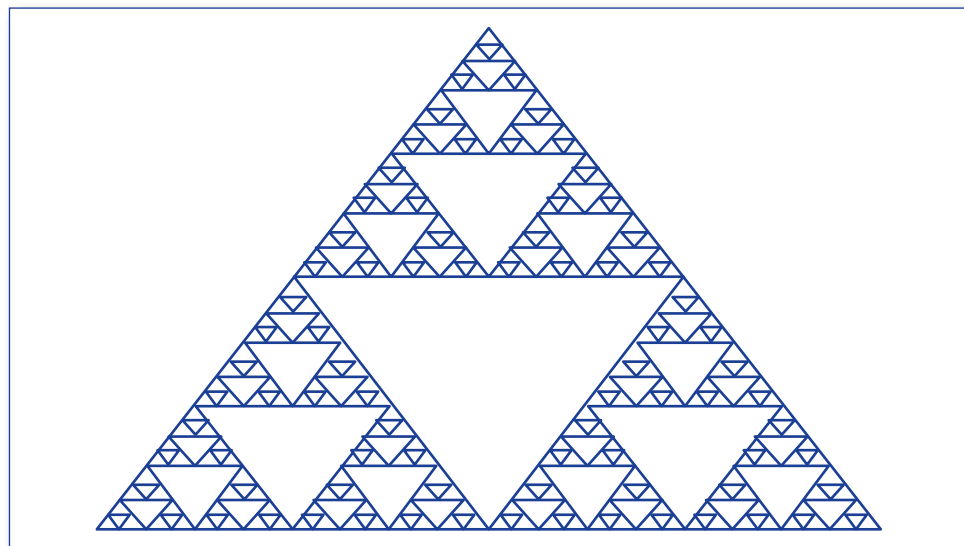
> Roughly speaking, fractals are complex geometric shapes with fine structure at arbitrarily small scales. Usually they have some degree of self-similarity. In other words, if we magnify a tiny part of a fractal, we will see features reminiscent of the whole. Sometimes the similarity is exact; more often it is only approximate or statistical. (Strogatz, 1994, p. 398)

Sierpinski's triangle (Figure 10.23) is one well-known fractal. Notice the self-similarity that exists within it. Strogatz noted that other fractals resemble naturally occurring objects such as clouds, coastlines, and blood vessel networks. Their ability to capture characteristics of natural objects, along with the fact that many discoveries in fractal geometry have been made in the very recent past (Devaney, 1998), make fractals a potentially exciting addition to school curricula.

Figure 10.22 Creating a simple Escher-like picture.

1. Cut a curve, corner to corner, from a shape that tiles the plane.

2. Translate the resulting object to the opposite side of the square.

3. Decorate your design as desired.

4. Use your design to tile the plane.

Figure 10.23 Sierpinski's triangle generated with an online applet (http://curvebank.calstatela.edu/sierpinski/sierpinski.htm).

Devaney (1998) described a game that can be used to introduce students to the mathematics underlying the construction of Sierpinski's triangle. He described the rules of the *chaos game* (Barnsley, 1989) in the following terms:

> First pick three points—the vertices of a triangle (any triangle works—right, equilateral, isosceles, whatever). Name one of the vertices 1,2, the second 3,4, and the third 5,6. The reason for these strange names is that we will use the roll of a die to determine the moves in the game. To begin the game, choose any point in the triangle. This point is the seed for the game. (Actually, the seed can be anywhere in the plane, even miles away from the triangle). Then roll a die. Move the seed halfway toward the named vertex: If 1 or 2 comes up, move the point half the distance to the vertex named 1,2. Now erase the original point and repeat this procedure, using the result of the previous roll as the seed for the next: Roll the die again to move the new point half the distance to the named vertex, and then erase the previous point. (p. 92)

After following the instructions above for a small number of rolls, students should connect the points generated. After many trials, regardless of the initial seed used, Sierpinski's triangle will begin to emerge from the pattern. The chaos game can be extended by asking students to experiment with changing the rules and observing the pattern generated, or by starting with a picture of a fractal and attempting to discover the rules that generated it.

CONCLUSION

Geometry presents many interesting ideas to study, dating from the historical era before Euclid to the present day. The fundamental goals of school geometry include understanding shapes and their definitions and constructing proofs. Currently, we know that far too few students completing high school geometry are able to write and understand proofs. Therefore, teachers must look for opportunities to help students move past naive empiricism and develop deductive reasoning. Dynamic geometry software can help progression toward this goal when used appropriately. As students move toward deductive proof, they should also develop an understanding of measurement as the process of iterating a given unit. The iteration process applies to the measurement of length, area, volume, and angles. Geoboards, rectangular grids, cubes, and wedges are among the physical tools helpful for developing students' conceptual understanding of measurement. Trigonometry takes angle measurement as one of its central objects of study, and at the same time presents an opportunity to further develop students' understanding of functions. Relatively recent developments in geometry related to transformations, tessellations, and chaos help add vibrancy to the subject. Sample four-column lessons are provided at the end of the chapter to further spark your thinking about teaching geometry.

VOCABULARY LIST

After reading this chapter, you should be able to offer reasonable definitions for the following ideas (listed in their order of first occurrence) and describe their relevance to teaching mathematics:

Euclid's *Elements*   307

Geometric habits of mind   308

Normative geometric thinking   308

van Hiele levels   311

HOMEWORK TASKS

1. Describe relationships that exist between the angles and arcs shown in Figure 10.3. Use dynamic geometry software and geometry textbooks as resources as necessary. Show all your work and justify your reasoning.

2. Suppose your school is searching for a DGE software program to adopt for its high school geometry courses. Examine the websites of at least three DGEs (e.g., Cabri, Geometer's Sketchpad, GeoGebra; do an Internet search to find the official websites). Then write a letter to the high school mathematics department chair recommending one of the software programs for adoption. Justify your adoption recommendation with details about the unique aspects of the selected DGE.

3. Use DGE software to construct a rhombus, a rectangle, and a kite that cannot be "messed up" (i.e., they retain their defining characteristics when dragged). Show and describe all steps in the construction. Then hide all portions of the construction extraneous to each shape, as done for the square on the right-hand side of Figure 10.7.

4. State a conjecture about the medians of a triangle (see Figure 10.9). Provide evidence that your conjecture may be true by giving paper-and-pencil examples that are carefully constructed or producing examples within a DGE. Then prove the conjecture deductively, drawing on other well-known results from geometry.

5. Determine how many segments of $\sqrt{10}$ can be found on a 10×10 geoboard. Also describe how to determine the number of segment lengths of $\sqrt{10}$ you can find on an $n \times n$ geoboard. Show all of your work and justify your answer.

6. Do an Internet search on "proofs of the Pythagorean theorem." Choose two proofs you would share with a high school geometry class you might teach in the future. Explain the essential similarities and differences between the logic of each of the proofs you choose.

7. Form a nonrectangular shape on a geoboard and describe how to determine its area. Your shape should be concave and have at least five sides. Then form another concave, rectangular shape on the geoboard and ask a classmate to find its area. The shape you give to your classmate should present a significant challenge.

8. Draw a diagram of a manipulative that would help students understand the idea of angle measurement in terms of the iteration of wedges. Write three progressively more difficult tasks that can be

solved by using the manipulative. Explain how your tasks could help students begin to understand and appreciate standard units of angle measure (e.g., degrees) and standard instruments for angle measurement (e.g., protractors).

9. Do an Internet search on "Mira geometry tool." Describe four different types of exercises that can be done using a Mira. Be sure to cite the website from which each exercise idea was generated.

10. Describe how students could use a clinometer to measure the heights of very tall objects, such as the school building or trees surrounding it. Write at least three exercises you would ask students to do outside the classroom using clinometers. Include a diagram showing proper use of the clinometer to measure a tall object.

CLINICAL TASKS

1. Download the University of Chicago's van Hiele geometry test (http://ucsmp.uchicago.edu/van_ Hiele.html) and administer it to a class of high school geometry students (be sure to request permission to do so, as noted on the project website). Score the test using one of the scoring schemes provided. Construct a data display that could be given to the teacher of the class to summarize the van Hiele levels of the students. Describe how the information in the data display could be used to guide decisions about instruction.

2. Download the University of Chicago's geometry proof test (http://ucsmp.uchicago.edu/van_Hiele .html) and administer it to a class of high school geometry students (be sure to request permission to do so, as noted on the project website). Score the test using one of the scoring schemes provided. Construct a data display that could be given to the teacher of the class to summarize the proof construction abilities of the students. Describe how the information in the data display could be used to guide decisions about instruction.

3. Interview at least three students. Ask each one to draw as many examples of trapezoids as he or she can. After the students have drawn several examples, ask them to write a definition for the word *trapezoid*. Describe what the students' drawings and definitions reveal about their personal prototypes for the concept.

4. Ask a student to use a DGE to construct a square that cannot be "messed up" when dragged (i.e., it remains a square even when portions of it are dragged). Describe how the student responds to the task. In your description, be sure to note whether a drawing or a construction is produced.

5. Ask a student to work with premade constructions in a DGE for a rhombus, a rectangle, and a kite. Ask the student to drag each shape and write about how it changes and stays the same under drag. After experimenting with each shape, have the student write a definition for each shape. Write a report that critiques the student's reasoning process and final definition. In your report, be sure to note whether the final definitions for each shape are hierarchical or partitional.

6. Interview a student who has completed high school geometry and provide him or her access to a ruler and protractor. Ask the student if it is true that all triangles have interior angles whose measures sum to 180 degrees. Then ask the student to justify his or her position. If the student does not use a deductive proof, ask whether he or she can produce a proof like those done in high school geometry classes. Write a report describing what you learned about the student's thinking. In your report, be sure to address whether or not the student exhibited an intellectual need to produce a deductive proof.

7. Interview a geometry teacher to determine which forms of proof he or she encourages students to use. In particular, ask if two-column, paragraph, or flowchart proofs are used. Try to determine why

the teacher uses the form(s) of proofs mentioned. Ask for specific examples of proof exercises students are required to complete. Write a report describing the forms of proof the teacher uses and the types of exercises students are expected to complete. Also give a personal reaction to the rationale he or she provides for choosing forms of proof for use in class.

8. Have a class of students complete the "broken ruler" problem shown in Figure 10.12. In looking through students' work on the item, note how many correctly answered the item and how many did not. Then analyze the papers of students who provided incorrect responses. Describe the type of mistake each student made. Then select two students who provided incorrect responses for interviews. Question the students to understand why they answered incorrectly. Write a report of your findings, along with a general strategy you would use to help students overcome misconceptions associated with this type of problem.

9. Interview three students. Ask each of them to evaluate the truth of the following statement: "As the perimeter of a closed shape increases, its area increases as well." Ask the students to justify their reasoning. Write a report on how the students evaluated the statement, including any diagrams or work they produced in the process of doing so. Then, based on what you learned about the students' reasoning, write a follow-up task you would ask them to solve. The follow-up task should be designed to help extend or correct the thinking patterns you observed.

10. Ask a student to determine the area of the trapezoid shown in Figure 10.16 without using a formula learned in school. Write a report that describes the strategies he or she used to determine its area. To prepare for the possibility that the student is not able to determine the area, write a set of hints you would use to prompt him or her in the right direction.

11. Ask a student to predict how many cubes it would take to fill the box shown in Figure 10.17. Also create a pattern picture of your own without an accompanying box picture and ask the student to predict how many cubes it would take to fill it if the pattern picture were folded into a box. Have the student check each prediction by using physical materials. Write a report that describes the student's initial predictions for each task and any revisions the student made to his or her conjectures after working with the physical materials.

12. Ask at least three students who have studied trigonometry if it is possible to have a situation where $\sin(x) = 2$. Also ask each student to determine the sign of $\cos(300°)$, (i.e., positive or negative). Finally, ask if $\sin(23°)$ is larger or smaller than $\sin(37°)$. Have each student explain his or her reasoning for each task. If students use a calculator, ask them if they can also solve each task without a calculator. Describe how each student responded to the tasks and then identify the conceptual and procedural elements implicit in their thinking.

13. Ask a class to do a task of your own design that requires a clinometer. Have them show all work involved in solving the task and then hand it in. Referring to the students' work, the level of challenge the tasks presented, and logistical issues, suggest ways to improve the activity the next time you use it.

14. Play the chaos game with a class of students. Split students into several small groups and have them record their results on clear overhead transparencies. Each transparency should have the three initial vertices for the game in the same location. Consider giving each group a movable dot so they do not have to erase points while playing the game. When the students are finished playing in small groups (let them do approximately 10 trials per group), overlay the transparencies on one another and note how closely they resemble a Sierpinski's triangle. Drawing on your experiences of playing the chaos game in class, take a position on the appropriateness of fractal geometry for middle and/or high school students. Defend your position by drawing on the observations you made while playing the chaos game with your students.

> **VIGNETTE ANALYSIS ACTIVITY**   **Focus on CCSS Standards for Mathematical Practice 1–8**

Items to Consider Before Reading the Vignette

1. Reread each of the CCSS Standards for Mathematical Practice in Appendix A. Which of these standards have you seen most often in classes you have observed? Which have you seen least often?

2. Provide a statement of the triangle inequality. Describe a strategy you could use to help students understand the inequality.

3. Suppose that in triangles ABC and DEF, side *AB* is congruent with side *DE* and side *AC* is congruent with side *DF*. We also know that the measure of angle A is greater than the measure of angle D. What can we conclude about the lengths of sides *BC* and *EF*? Why?

4. Item 3 above suggests a geometric theorem commonly called the "hinge theorem." Why do you think it has this name? How would you help students understand the connection between the name of the theorem and its content?

Scenario

Mr. Martz was just beginning his student teaching semester. He had serious misgivings about embarking on a career in teaching. During his classroom observations, he had noticed that many students did not have what he considered "basic skills," such as the ability to solve simple equations and to factor and multiply polynomials. They also seemed indifferent about studying mathematics, whereas he loved the subject. Now he would be responsible for teaching the students he observed. Nervous about how he would be effective as their classroom teacher, he had frequent conversations with his mentor teacher and university supervisor about strategies he could employ. Mr. Martz was not completely convinced that the strategies they suggested would work, but nonetheless did try to take some of their advice into account. In one of the first geometry lessons he was responsible for teaching, students were to learn the geometric "hinge theorem" (see items 2 and 3 in the previous section). In teaching the lesson, Mr. Martz relied on a combination of his own intuition about students and the advice he received from his mentor and university supervisor.

The Lesson

The lesson began with a warm-up activity intended to review the triangle inequality, an idea taught the previous day. Mr. Martz put three sets of segment measurements on the board and asked students to determine whether or not they would form a triangle. The sets were the following:

    a. 1, 5, 7
    b. 3, 6, 8
    c. 4, 3, 7

In part a, students were to notice that $5 + 1 < 7$, so the sides could not form a triangle. In part b, they were to see that any combination of two side lengths added together would be greater than the

remaining side length. Therefore, the side lengths in part b would form a triangle. In part c, students needed to reason that since $4 + 3 = 7$, the segments could not form a triangle. Mr. Martz felt these three cases would be adequate for reviewing the main aspects of the triangle inequality.

After about five minutes of socializing with one another, students settled in and began to write the warm-up activity in their notebooks. As Mr. Martz circulated about the room, he was surprised to see that many did not know how to start. As he answered students' questions, he felt as if he were reteaching the previous day's lesson on an individual basis to each of the 25 students in the classroom. He was especially disheartened because the previous day's lesson had involved using a concrete manipulative, popsicle sticks, to teach the triangle inequality. This strategy was recommended by both his university supervisor and mentor teacher. Sensing Mr. Martz's frustration, his mentor teacher asked the class to stop working on the warm-up problems and direct their attention to her. The mentor teacher reminded students of the previous day's popsicle stick activity and asked students to think of the segment lengths in terms of the popsicle sticks. Questions about how to start the activity then subsided as students seemed to connect the work done with the popsicle sticks to the task at hand. Some students requested rulers so they could draw the popsicle sticks to scale to solve the review exercises. Within 10 minutes, students were ready to move on.

Next, Mr. Martz started what he considered to be the main part of the lesson: teaching the hinge theorem. He asked students to pair up. After milling about for a couple of minutes, each student seemed to have found a partner. The pairs of students were directed to use protractors to obtain two angles. One pair member was to produce a 50° angle, and the other was to produce a 30° angle. Popsicle sticks were to be used to indicate the side lengths in each angle. After the angles had been formed with the popsicle sticks, students were to measure the distance from the tip of one stick to the other in each angle. Mr. Martz had originally planned to have students do these measuring and constructing activities on their own, but now decided to show students how to do each step in the process at the document camera in front of the room, fearing that students left to their own devices might do some steps incorrectly. When Mr. Martz finished demonstrating the steps in the activity, he told students to notice that the distance between popsicle stick tips was greater for the 50° angle than for the 30° angle.

After doing the popsicle stick demonstration for the class, Mr. Martz asked students to break up from their pairs and return to individual work. Students noisily gathered their protractors, popsicle sticks, and pencils to return to their original seats. When most students had settled in once again, Mr. Martz distributed a set of class work exercises. In each exercise, the students were given a pair of triangles. In each triangle, the length of two sides and the measure of their included angle was given. From this information, students were to determine whether the length of the nonincluded side was greater in the first triangle or the second. Mr. Martz showed students how to do the first exercise and then directed them to finish the remaining exercises on their own.

Within 10 minutes, all students had either finished or stopped doing the class work problems and begun to socialize. A few had their heads down and were sleeping. Mr. Martz had not anticipated that his students would finish this portion of the lesson so quickly. He considered starting the next day's lesson, but there were only seven minutes left in the class period. Instead of starting something new, he opted to give them their homework assignment. Mr. Martz loudly announced the page number and exercise numbers for the homework. Some students wrote the information down while others continued to talk. A few opened their books to begin the assignment, but most kept talking. Mr. Martz had a sinking feeling that his lesson had ended with a thud.

Questions for Reflection and Discussion

1. Which CCSS Standards for Mathematical Practice are most evident in the vignette? Which are least evident? In regard to those that are least evident, how could the lesson be improved to help students better work toward the standards?

2. Comment on the numbers Mr. Martz chose for the warm-up exercises. Is his number choice helpful for bringing out the main aspects of the triangle inequality? What improvements could be made?

3. How could the review of the triangle inequality have gotten off to a smoother start? Suggest specific steps to be taken.

4. Comment on Mr. Martz's use and organization of group/pair work. Were there times he should have used it and did not? How could the process of forming groups be made more effective and efficient? What kinds of tasks would be meaningful for pairs or groups to do in the context of learning the hinge theorem?

5. How could Mr. Martz get students more involved in discovering the hinge theorem?

6. Did the structure of the lesson give Mr. Martz good opportunities to assess students' geometric understanding? Why or why not?

7. Would dynamic geometry software be helpful for enhancing any portions of this lesson? Why or why not?

RESOURCES TO EXPLORE

Books

Albrecht, M. R., Burke, M. J., Ellis, W., Kennedy, D., & Maletsky, E. (2005). *Navigating through measurement in Grades 9–12.* Reston, VA: NCTM.

Description: The authors provide a collection of activities for teaching measurement in an inquiry-oriented manner to high school students. Activities address the process of measurement, using formulas to measure complex shapes, discovering and creating measurement formulas, and measuring with technology.

Battista, M. T. (2003). *Shape Makers.* Emeryville, CA: Key Curriculum Press.

Description: This book supports students' work with Shape Makers in the Geometer's Sketchpad dynamic geometry environment. As students manipulate preconstructed shapes, they develop deeper understanding of a hierarchy of quadrilaterals.

Clements, D. H. (Ed.). (2003). *Learning and teaching measurement* (Sixty-fifth yearbook of the National Council of Teachers of Mathematics). Reston, VA: NCTM.

Description: This yearbook contains a number of articles relevant to teaching and learning measurement in secondary school. Articles for secondary school teachers focus on estimating areas of irregular shapes, exploring measurement through literature, and using geoboards to teach measurement.

Craine, T. V. (Ed.). (2009). *Understanding geometry for a changing world* (Seventy-first yearbook of the National Council of Teachers of Mathematics). Reston, VA: NCTM.

Description: This yearbook consists of a collection of articles useful for teaching various aspects of geometry. Articles address topics such as teaching geometry for conceptual understanding, using interactive geometry software, having students discover geometric theorems, and exploring fractals in nature.

Day, R., Kelley, P., Krussel, L., Lott, J .W., & Hirstein, J. (2002). *Navigating through geometry in Grades 9–12.* Reston, VA: NCTM.

Description: The authors provide activities that can be used to teach geometry in an inquiry-oriented manner to high school students. Activities address geometric transformations, similarities, and fractals.

Pugalee, D. K. Frykholm, J., Johnson, A., Slovin, H., Malloy, C., & Preston, R. (2002). *Navigating through geometry in Grades 6–8.* Reston, VA: NCTM.

Description: The authors provide activities that can be used to teach geometry in an inquiry-oriented manner to middle school students. Activities address characteristics of shapes, coordinate geometry, transformations, and visualization.

Websites

Learning Math: Geometry: **http://www.learner.org/resources/series167.html#program_descriptions**

Description: This website contains videos relevant to teaching geometry in the middle school. Programs of interest to middle school teachers deal with proof, the Pythagorean theorem, similarity, and solids.

van Hiele Levels and Achievement in Secondary School Geometry: **http://ucsmp.uchicago.edu/resources/van-hiele/**

Description: This website describes the work of the Cognitive Development and Achievement in Secondary School geometry project. It provides insight on a large sample of secondary students' understanding of geometry, and it also contains tests that can be used to assess students' van Hiele levels and proof writing abilities.

FOUR-COLUMN LESSON PLANS TO HELP DEVELOP STUDENTS' GEOMETRIC THINKING

Lesson Plan 1

Based on the following NCTM resource: Groth, R. E. (2006). Expanding teachers' understanding of geometric definition: The case of the trapezoid. *Teaching Children Mathematics, 12,* 376–380.

Primary objective: To help students understand two commonly accepted definitions of *trapezoid* and the consequences of each for quadrilateral classification schemes.

Materials needed: Paper, pencil, chalkboard, computers with Internet access

Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
1. Put the following writing prompt on a screen or chalkboard at the beginning of the lesson: "Give an example of a word that has more than one definition. The word does not have to be from mathematics. Write at least two different definitions for it."	Students may list words whose different meanings can be determined from the context in which they are used (e.g., *hack* can mean to physically strike something or to break into a computer). Some may list words that can be either nouns or verbs (e.g., *storm*).	Emphasize that words can often be assigned a variety of definitions. Mention that even some words in mathematics, such as *trapezoid*, can take on different meanings in different definitional systems.	After students have written for a few minutes, have some of them share their responses with the rest of the class. Ask students for examples of mathematical words that can be defined in different ways. This will help assess whether or not they believe that only one definition can be "right" in mathematics.
2. Have students work in small groups to produce as many examples of trapezoids as possible. After working for a few minutes, each group should send a representative to the chalkboard to post the examples they generated.	Some groups will restrict their examples to quadrilaterals with *exactly* one pair of parallel sides.

Other groups will include examples of quadrilaterals with *at least* one pair of parallel sides.

Some groups will produce shapes that are not quadrilaterals. | At this point in the lesson, do not censor the shapes on the chalkboard. The purpose of this portion of the lesson is to engage students in brainstorming, not to formalize their thinking. The shapes on the board will serve as a catalyst for discussions throughout the rest of the lesson. | Ask students which shapes on the board should be considered trapezoids and which should not. This should help elicit students' current ideas about how trapezoids are to be defined. It will also get students thinking about how they might formally define a trapezoid so that the definition includes shapes they believe to be trapezoids. |

(Continued)

Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
3. Have students return to small-group work. Give them the task of writing a formal definition for the word *trapezoid* based on the discussion and on examples that have been given up to this point.	Some groups will write definitions with more detail than necessary. Others will not include enough detail to guide the reader to produce an example of something they would consider a trapezoid.	Draw attention to weaknesses in students' definitions. For those that include too much detail, use just the necessary components of the definition to produce examples. For those that do not include enough detail, produce examples that are based on the definition to highlight the inadequacy.	Ask students to write second drafts of their group definitions based on the class discussion that occurred after the first drafts were shared. The second drafts will become their working definitions for the next step in the lesson.
4. In groups, students should do Internet searches on "define trapezoid." Have each group use a different search engine (e.g., Google, Yahoo!, Bing) so they obtain slightly different results. They should be prepared to share their results with the rest of the class.	Students will likely find three types of definitions for *trapezoid*: a quadrilateral with exactly one pair of parallel sides, a quadrilateral with at least one pair of parallel sides, and nonmathematical definitions.	Ask each group to report the definitions they found. After they have reported, point out the three types of definitions. Emphasize the two mathematical types of definitions as being commonly used in mathematics curriculum materials.	As students report on the definitions they have found, assess whether or not they found the two types of mathematical definitions. If they did not find both types, be sure to introduce the missing type of definition into the class conversation.
5. Based on the discussion of definitions found online, have students once again produce as many examples of trapezoids as possible.	Some will still resist the idea that more than one legitimate definition can exist, and others will use both definitions to produce examples.	Emphasize the importance of consistency within a system rather than one absolutely "correct" definition for every possible system.	Give a writing exercise: "Can squares, rhombi, and parallelograms ever be considered trapezoids? Why or why not?"

How This Lesson Meets Quality Control Criteria

- *Addressing students' preconceptions:* This lesson connects to school-based and experiential knowledge of defining concepts and expands on students' previous knowledge of geometry by introducing the idea of the existence of more than one legitimate definition.
- *Conceptual and procedural knowledge development:* The lesson addresses the concept that different consequences follow different definitions of a shape and introduces two different definitions that essentially specify procedures to produce trapezoids.
- *Metacognition:* Students are encouraged to compare their thinking with that of classmates at several different points in the lesson, particularly during group discussion. Students are encouraged to compare their thinking to information found online.

Lesson Plan 2

Based on the following NCTM resource: Britton, B. J., & Stump, S. L. (2001). Unexpected riches from a geoboard quadrilateral activity. *Mathematics Teaching in the Middle School, 6,* 490–493.

Primary objective: To help students develop and use strategies for sorting quadrilaterals into families

Materials needed: One geoboard for each student, one piece of dot paper for each student

Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
1. Have students work in groups and attempt to find all possible quadrilaterals that can be formed on a 3 × 3 section of a geoboard. They should keep track of the quadrilaterals they find by sketching them on a piece of dot paper.	Students will discuss what it means to have different quadrilaterals. Some will interpret "different" to mean noncongruent. Others will think that two shapes that are the same except for size are not different.	While working with groups, encourage them to think of "different" as meaning "noncongruent." Congruence and noncongruence can be determined by laying one shape on top of another and trying to line the two up.	Ask students if a large square and a smaller square can be considered "different." If they do understand that the two squares are considered different in the context of this activity, they are ready to engage fully in the lesson.
2. Have groups share the results of their geoboard explorations. Begin by having each group share one quadrilateral they formed. Continue having groups share one quadrilateral at a time until they have no new different ones to share.	In some cases, groups will present shapes that are congruent to a shape already presented. Groups may not consider the possibility of having concave quadrilaterals.	When groups present shapes congruent to those already presented, ask if a transformation could be performed on the shape to make it the same as another. If concave quadrilaterals are not presented, show an example of one and have students look for the rest of the possible concave quadrilaterals.	Keep track of all examples presented to determine if students have identified all 16 possible quadrilaterals. In classes where some of the quadrilaterals are not identified, provide hints that will guide students to discover them (as with the concave quadrilaterals mentioned in the cell to the left).
3. Take all 16 possible geoboard quadrilaterals and sort them into two groups. In one group, place all shapes that have an obtuse interior angle. Place the rest of the shapes in a second group. Tell students that all of the quadrilaterals in one group have an attribute that is shared by none of the quadrilaterals in the other group. Ask them to identify the attribute.	Some students may immediately notice that obtuse angles set one group apart from the other. Some may have difficulty expressing their thoughts in formal geometric language. Some may identify attributes that are actually shared by both groups.	If a student immediately identifies obtuse angles as the relevant attribute, do not immediately comment on the correctness of the answer. Instead, encourage the class to look for additional possible differentiating attributes. When students use informal language (e.g., "pointy" rather than "acute"), introduce the corresponding formal term.	When students have difficulty expressing their thinking in terms of formal geometric language, look for opportunities during the discussion to assess their acquisition of the language. For example, students who originally do not use the formal term *acute* should be asked to provide descriptions of shapes that contain acute angles at various points in the class discussion.

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Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
4. Give students a worksheet that shows all 16 possible quadrilaterals on dotpaper. Have students cut the shapes apart and classify them in at least two different ways. They should be prepared to present their classification schemes to the rest of the class.	Students may use various different properties to form different classifications, including: number of sets of parallel sides, number of right angles, and type of symmetry. For example, some may put all quadrilaterals with one right angle into one group and the rest of them into another.	Encourage students to go beyond the stated requirements of the task by finding as many ways as possible to sort the quadrilaterals. Encourage and support their use of formal language in describing their categorization ideas to one another.	Assess whether or not students use classification schemes different from those discussed during class. In cases where students do not come up with their own original categorization schemes, prompt them to devise some.
5. As an extension to the main activity for the day, encourage students who finish early to determine the areas of the 16 different shapes introduced during the lesson.	Some students will be eager to take on the task, having exhausted interesting ways to categorize the shapes. Others will need time to continue to categorize the shapes.	Encourage students who work on the area task to share their strategies with one another. Time permitting, choose a few students to present strategies to the entire class.	As students present their area measurement strategies, look for evidence of original thought rather than mere use of previously learned formulas for determining area.

How This Lesson Meets Quality Control Criteria

- *Addressing students' preconceptions:* The informal language and sorting strategies students have learned outside of school are connected to formal language and sorting strategies commonly used in geometry.
- *Conceptual and procedural knowledge development:* Procedurally, students encounter definitions for different geometric shapes; conceptually, they come to understand and appreciate the thinking processes involved in sorting and defining.
- *Metacognition:* Students are asked to examine their thinking to determine whether or not they have produced all possible shapes on a 3×3 section of a geoboard; they are asked to examine their use of the word different in reference to shapes.

Lesson Plan 3

Based on the following NCTM resource: Kaufmann, M. L., Bomer, M. A., & Powell, N. N. (2009). Want to play geometry? *Mathematics Teacher, 103,* 190–195.

Primary objective: To help students understand and appreciate the axiomatic structure of geometry

Materials needed: A sheet of poster board for each group of students, markers, household objects to be used for games (see Step 1)

Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
1. Students should be divided into groups of two to four each. Give each group a miscellaneous set of household objects (e.g., buttons, pins, egg cartons, dice, balls, marbles). Tell each group to devise a game that uses the objects and consists of at least five rules.	Students will be able to draw on their out-of-school knowledge of games to construct rules. However, for some games, it is likely that some rules may contradict each other. It may also be that some rules are not complete. Some rules may also be repetitive.	Allow students to work freely in groups at this point in the lesson. Do not intervene to correct them at this point, since the next step of the lesson involves peer review. One of the objectives of peer review is to develop skill at noticing possible mistakes.	Observe and listen to students as they work and take note of which games have rules that are contradictory, incomplete, or repetitive. Also note the specific flawed rules. This information will be needed later in the lesson.
2. Tell students that they will be reviewing one another's games before they will be marketed. Lead a whole-class discussion about how to determine if the rules for a game are reasonable.	Some students will give ideas that correspond to the three main categories of interest: contradictory, incomplete, and repetitive rules. Others will give ideas not related to these categories.	List all student ideas on the board or a screen as they are given. Near the end of the discussion, highlight the student suggestions that correspond to the three main categories of interest.	Monitor student contributions for evidence of suggestions aligning with the three categories of interest. Encourage further discussion until all three categories have arisen.
3. Have each group construct a poster to display the title of their game and its rules. Posters will then be sent out for review by classmates.	Since the previous portion of the lesson dealt with identifying contradictory, incomplete, and repetitive rules, examples of such rules should be identified by students.	Draw upon the assessment information gained in Step 1 in the lesson to try to ensure that students are not missing important flaws in the games they are reviewing. Draw attention to flaws that students do not identify if they incorrectly believe they have spotted them all.	Assess the students' critiques against the observations you made in assessing Step 1. If students missed a substantial number of flaws while doing peer reviews, choose sample games to critique together during whole-class discussion.
4. Build the analogy between the games the students constructed and axiomatic systems. To do this, ask questions such	Students with some previous knowledge of geometric proof may connect the rules of a game to axioms, and	Scaffold students' learning by asking for specific examples of if-then statements, conjunctions, and the	By listening to students' responses, assess whether or not they have difficulty understanding the role of if–then statements or the

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Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
as the following: "What role do the rules play?" "What role do if–then statements play?" "What happens if you change conjunctions like *and* or *or*?" "What happens if you remove one of the rules?" "What elements of the game correspond to geometric theorems and undefined terms?"	understand that removing one rule changes the system substantially. These students may also speak of objects used to play the game as undefined terms, and of plays that occur during the game as theorems. Even those without much proof experience may recognize the role of if-then statements and how a rule is often changed substantially when a conjunction is changed.	effect of removing a rule from the game or changing a conjunction. Have them draw examples from their own games or from the games they reviewed. Introduce formal geometric language as necessary if it does not arise in the conversation.	impact of changing a conjunction. If so, have them play the game again, this time with one of the rules containing an if–then statement or conjunction changed. They can then report back on their observations.
5. Ask students to compare the U.S. government to an axiomatic system. Lead a brainstorming session about how the U.S. government is axiomatically different from other forms of government.	Students will offer various ideas. They may compare the U.S. Constitution to a set of postulates, compare legislation to theorems, and give reasons for adhering to different axiomatic systems.	Record student ideas on the board as they are offered. Encourage students to construct analogies about postulates, theorems, and axiomatic systems if they do not arise naturally.	Assess the strengths and weaknesses of the analogies students offer. In cases where analogies are greatly stretched, ask if a different analogy that uses the same formal terminology can be constructed.

How This Lesson Meets Quality Control Criteria

- *Addressing students' preconceptions:* Students' out-of-school experiences with rules of games are drawn on to build the idea of an axiomatic system. Students' school-based knowledge of government provides another analogous situation for study.
- *Conceptual and procedural knowledge development:* The activities develop the overall concept of an axiomatic system. Students work with the set of permissible procedures within the systems they investigate.
- *Metacognition:* After receiving classmates' critiques, students are prompted to reexamine the rules they established for the games at the beginning of class. Students reexamine the rules they established by comparing them to geometric axioms.

Lesson Plan 4

Based on the following NCTM resource: Buhl, D., Oursland, M., & Finco, K. (2003). The legend of Paul Bunyan: An exploration in measurement. *Mathematics Teaching in the Middle School, 8,* 441–448.

Primary objective: To model length, area, and volume measurement and connect them to scale factor and proportion

Materials needed: Cardboard replica of an ax; dimensions of length, breadth, average depth, maximum depth, and volume for a local lake; modeling clay for student use; a piece of grid paper for each student; a set of cubes for each student

Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
1. Read a version of the legend of Paul Bunyan to the class (see www .paulbunyantrail.com/). Be sure to emphasize (1) the size of Paul Bunyan's blue ox, Babe; (2) the dimensions of Paul Bunyan's skillet; and (3) the volume of the lake in which Paul Bunyan worked. After reading the story, ask students to identify connections to geometric measurement.	Students may identify length, area, and volume as important elements in the story. These elements pertain to Babe's height, Paul's skillet, and Paul's length, respectively. Students may identify additional elements in the story that may be measured.	List all student responses as they are offered. To summarize their responses and lead into the next portion of the lesson, emphasize ideas that include measurement of length, area, and volume. Acknowledge the validity of geometric measurement ideas that fall outside these three aspects as well.	As students offer ideas, start asking which units are commonly used to measure length, area, and volume. Also ask them to explain why these measures are used. Students' responses will begin to provide a sense of whether they understand geometric measurement conceptually.
2. Show students a cardboard replica of an ax. Mention that the distance between Babe's horns was 42 ax handles. Then ask students to estimate the following lengths in terms of ax handles and also in terms of the distance between Babe's horns: (1) the length of the school building and (2) the distance from school to home.	NAEP data show that students often have difficulty measuring one distance in terms of another. Students may want to use a standard unit of measurement instead of trying to estimate lengths using nonstandard units of measure.	Remind students that length measurement, whether using standard or nonstandard units, consists of iterating the units end to end so there are no gaps or overlaps. If they struggle to begin the tasks, have them take the cardboard ax and measure out 42 ax handles to understand the size of the unit of measure being used.	Assess students' work for reasonable estimates at this point. It is not necessary that they know the exact length of the school building or the exact distance from school to home. However, before moving on to the next few tasks, make sure their estimates reflect reasonable approximations.
3. Tell students that Paul Bunyan's skillet covered an acre of land (43,560 ft^2). Ask them to determine its radius.	Setting up the proportion incorrectly can produce an answer of 50,824 feet for the radius.	If students obtain 50,824 feet, ask them to draw a diagram of a circle with such a radius and determine its area.	Ask students who obtain 50,824 feet for the radius why this cannot be a correct answer.
4. Tell students that Paul Bunyan was supposed to have created lakes by stomping through muddy land. The students' task will be to examine data regarding the dimensions of	Students may take several factors into account, including the possible dimensions of Paul's foot, the amount one would expect a	Encourage students to think about the dimensions of a foot in terms of area. If necessary, have students trace out a foot on grid paper and then think	A broad range of answers are possible, but the key thing to assess is whether or not the answers are justified by mathematics and the context of the story.

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Steps of the lesson	Expected student responses	Teacher's responses to students	Assessment strategies and goals
a lake to argue whether or not this is a reasonable extension to the legend. Choose a local lake and provide data on its length, breadth, average depth, maximum depth, and volume. For dimensions of the Great Lakes, do an Internet search on "Great Lakes dimensions."	foot to sink in mud, and whether or not the dimensions of Paul's foot would line up with the dimensions of a lake. They might also consider whether or not several footprints end to end would measure out the given dimensions of a lake.	about how many pieces of grid paper would be needed for Paul's foot. If it is necessary to help students think about the reasonableness of volume measurement, have cubes available that they can use to create a scale model of the chosen lake.	Students may come to different conclusions about whether or not it is a reasonable extension of the legend to say that Paul created lakes with his footprints, but their justifications must be reasonable.
5. To conclude the activity, have students use modeling clay to construct scale models of several of the objects of the lesson, including Paul, Babe, and Paul's frying pan.	Students will draw on measurements done earlier in the lesson to construct Paul and Babe. The frying pan may be constructed as a square and then molded into a circle. Some students will finish early.	Ask students who finish the task before others to construct a scale model of the lake explored earlier. Those still working can check their models by comparing them with one another.	At the end of the lesson, have students share their scale models with the rest of the class. As they present, check to see whether the dimensions they have chosen are reasonable.

How This Lesson Meets Quality Control Criteria

- *Addressing students' preconceptions:* Students' out-of-school experiences with legends and tall tales are engaged as they evaluate the sizes of objects from one particular tale. Students begin to make connections among the previously learned concepts of distance, area, volume, ratio, and proportion.
- *Conceptual and procedural knowledge development:* Students are prompted to explore the concept of measurement in terms of the foundational principle of iteration of a unit. Students draw on procedural knowledge in solving proportions.
- *Metacognition:* Students are prompted to evaluate the reasonableness of their measurements by checking their results against the description of events provided in the legend. The teacher plays a role in prompting students to rethink their estimations when they are not reasonable.