CHAPTER 2. SPECIFICATION OF SIMULTANEOUS EQUATION MODELS

In model specification, the researcher uses prior theory to detail a series of equations and represent these using path models, equations, and/or matrices. Simultaneous equation models contain random variables (i.e., observed variables and error terms) and structural parameters (i.e., constants providing intercepts and the relationships between variables). The variables of a simultaneous equation model may be linked through direct relationships, indirect relationships, reciprocal relationships, feedback loops, and/or correlations between disturbances. Theory plays an instrumental role in model creation and determines the theoretical, or structural, relationships between the variables of interest. Empirical methods assess the fit of these specified models to the data, as we will explain later.

The general matrix representation of simultaneous equation models appears in Equation 2.1:

$$\mathbf{y} = \mathbf{B}\mathbf{y} + \mathbf{\Gamma}\mathbf{x} + \boldsymbol{\zeta} \tag{2.1}$$

Endogenous variables, denoted by **y**, are outcome variables or variables determined within the model. The vector of endogenous variables has dimensions $p \times 1$. Exogenous variables, denoted with the $q \times 1$ vector, **x**, are exogenous variables in the model (i.e., they are not explained by the model). For computational ease, random variables are assumed to be deviated from their means. Disturbances, or errors in the equations, are represented with $\boldsymbol{\zeta}$, a $p \times 1$ vector. There is one disturbance per endogenous variable, hence the similar dimensions. Gamma coefficients (γ) describe the effect of exogenous variables on the endogenous variables and are summarized in the coefficient matrix $\mathbf{\Gamma}$, which has dimensions $p \times q$. Beta coefficients (β) describe the effect of an endogenous variable on another endogenous variable. They are summarized in the coefficient matrix $\mathbf{B}$, with dimensions $p \times p$. Table 2.1 summarizes the components of a simultaneous equation model, including vector/matrix names, definitions, and dimensions (Bollen, 1989b; Kaplan, 2009).

Also in Table 2.1, two additional covariance matrices are important in describing simultaneous equation models. The matrix designated by $\mathbf{\Phi}$ is the variance/covariance matrix of the exogenous variables (xs), and $\mathbf{\Psi}$ is the covariance matrix of the disturbance terms (ζs). Both covariance matrices are symmetric.

Table 2.1 Notation for Simultaneous Equation Models

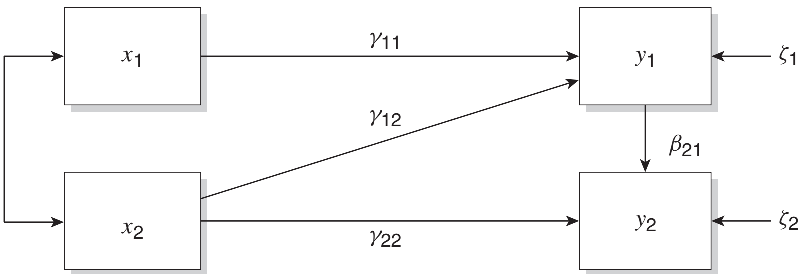
Vector/Matrix	*Definition*	*Dimensions*
Variables		
$\mathbf{y}$	Endogenous variables	$p \times 1$
$\mathbf{x}$	Exogenous variables	$q \times 1$
$\boldsymbol{\zeta}$	Disturbance terms or errors in equations	$p \times 1$
Coefficients		
$\boldsymbol{\Gamma}$	Coefficient matrix for the exogenous variables; the effect of exogenous variables on endogenous variables; direct effects of x on y	$p \times q$
$\mathbf{B}$	Coefficient matrix for the endogenous variables; the effect of endogenous variables on endogenous variables; direct effects of y on y	$p \times p$
Covariance matrices		
$\boldsymbol{\Phi}$	Covariance matrix of exogenous variables, x	$q \times q$
$\boldsymbol{\Psi}$	Covariance matrix of disturbance terms, ζ	$p \times p$

A number of assumptions apply to these models. Simultaneous equation models assume that the endogenous and exogenous variables are directly measured and have no measurement error. The disturbances include all variables influencing y that are omitted from the equation and are assumed to have expected values of zero ($E(\boldsymbol{\zeta}) = 0$). Disturbances are further assumed to be uncorrelated with the exogenous variables, homoscedastic and nonautocorrelated. Violations are possible; some will be treated in this monograph, while others are addressed elsewhere (e.g., Kmenta, 1997). The random variables are assumed not to have instantaneous effects on themselves.

Path Diagrams, Equations, Matrices: An Example of Specification

A path diagram pictorially represents how variables are related to one another in a theoretical model. Path diagrams use particular conventions: Variables shown in rectangles are observed variables, single-headed arrows denote the direction of influence, and double-headed arrows depict a covariance not explained in the model. Errors in the equations could technically be placed in ovals (representing unobserved or latent variables), but they are typically depicted unenclosed. Figure 2.1 is a path diagram of a recursive model containing two exogenous and two endogenous variables. Following the notation introduced above, the gamma coefficients represent direct effects of exogenous (x) variables on endogenous (y) variables: γ_{11} is the coefficient for the path from x_1 to y_1, γ_{12} is the coefficient for the path from x_2 to y_1, and γ_{22} is the coefficient for the path from x_2 to y_2.[1] β_{21} is the coefficient for the path from y_1 to y_2.[2]

Figure 2.1 Path Diagram of a Recursive Model



The model can be written as a series of equations, one for each endogenous variable. Two equations correspond to the path diagram in Figure 2.1 (intercepts are unnecessary, given that the random variables are assumed to be deviated from their means).

$$y_1 = \gamma_{11}x_1 + \gamma_{12}x_2 + \zeta_1 \tag{2.2}$$

$$y_2 = \beta_{21}y_1 + \gamma_{22}x_2 + \zeta_2 \tag{2.3}$$

The two representations are equivalent, although the equations do not provide the important information that ζ_1 and ζ_2 are uncorrelated.

[1]Coefficient subscripts follow particular conventions: The first number in the subscript denotes the variable being influenced, and the second number is the variable doing the influencing. Subscripts also denote placement in the coefficient matrix, row and column positions.

[2]The path with two-headed arrows between the exogenous variables x_1 and x_2 represents the observed covariance between these two variables. Covariances are generally represented as two-headed arrows: One convention is to use a curved path, and another convention is to use a straight line with multiple-headed arrows.

The same model can finally be written as a matrix equation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta_{21} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \tag{2.4}$$

With two endogenous variables in the model, the **y** vector has dimensions 2×1. The **B** matrix, the coefficient matrix for endogenous variables, includes β_{21}, the effect of y_1 on y_2. The $\mathbf{\Gamma}$ matrix contains coefficients for the effects of exogenous variables on endogenous variables and has dimensions 2×2. The two disturbance terms are shown in $\boldsymbol{\zeta}$, written as a vector with dimensions 2×1.

The variances of the exogenous variables and the assumed covariance between them are shown in the $\mathbf{\Phi}$ matrix with dimensions 2×2.[3] We follow a common convention of only displaying the lower diagonal of symmetric matrices. The $\boldsymbol{\Psi}$ matrix, which includes variances of the errors in the equations, has dimensions 2×2. In this model, the $\boldsymbol{\Psi}$ matrix is diagonal—the disturbances are not correlated.

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{11} & \\ 0 & \psi_{22} \end{bmatrix} \tag{2.5}$$

The matrix equations reveal certain properties of simultaneous equation models. For instance, variables do not have instantaneous effects on themselves, as shown in the zeros down the diagonal of the **B** matrix.

From Theory to Models: The Implied Covariance Matrix

Understanding simultaneous equation models is aided by switching one's frame of reference to a focus on the covariances among variables rather than individual cases in a sample (Bollen, 1989b).[4] The focus is on the fundamental statistical hypothesis,

$$\mathbf{\Sigma} = \mathbf{\Sigma}(\boldsymbol{\theta}) \tag{2.6}$$

[3]Unless a researcher is working with experimental data, in which zero correlation between exogenous variables can be assured, it is typical not to assume zero correlation.

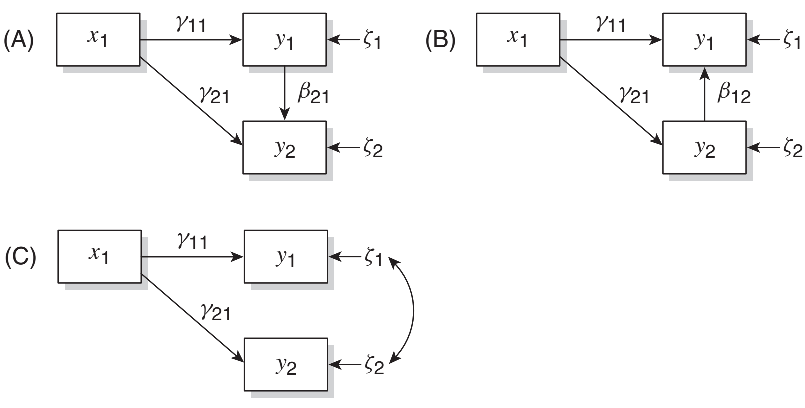
[4]Certainly individual observations matter a great deal and cannot be ignored. For example, outliers may influence the results of a given analysis.

where $\mathbf{\Sigma}$ is the population covariance matrix of the observed variables, $\boldsymbol{\theta}$ is a vector of the parameters to be estimated, and $\mathbf{\Sigma(\theta)}$ is the covariance matrix implied by your model (written as a function of the model parameters). In layman's terms, Equation 2.6 equates "your data" (at the population level) and "your model," which is precisely what conventional statistical techniques are designed to do. The fundamental statistical hypothesis underlies all aspects of modeling in simultaneous equation models: specification, identification, estimation, and assessment.

To understand how $\mathbf{\Sigma} = \mathbf{\Sigma}(\boldsymbol{\theta})$ relates to specification, it is useful to take a step back and view the big picture. Researchers typically have a set of variables in which they are interested, and they have some model in mind of how these variables fit together. This model may appear in a set of equations or as a path diagram.

The bulk of information needed for specification and estimation is summarized in the variances and covariances between the observed variables. That is, the total raw association between the variables is captured in the matrix $\mathbf{\Sigma}$ and is known at least in the population.

Figure 2.2 Three Models From the Same Variables



A researcher's model makes the argument that the association between variables is due to the hypothesized structure. Consider the three models displayed in Figure 2.2, Panels A, B, and C. Each model in Figure 2.2 uses the same three variables: two endogenous variables (y_1 and y_2) and one exogenous variable (x_1). But each is configured differently.

In Figure 2.2, Panel A, the endogenous variables, y_1 and y_2, are mutually dependent on the exogenous variable, x_1, and y_1 influences y_2. In Figure 2.2, Panel B, the two endogenous variables, y_1 and y_2, are again mutually dependent on the exogenous variable, x_1, but here y_2 influences y_1. In Figure 2.2, Panel C, the endogenous variables, y_1 and y_2, are mutually dependent on the single exogenous variable x_1. They are also related through an unexplained association between their errors. Theory drives which model a researcher would choose.

Creating the implied covariance matrix, $\mathbf{\Sigma(\theta)}$, allows a researcher to break down exactly how a hypothesized model relates to the known variances and covariances among the observed variables. As an example, consider the covariance between x_1 and y_1: $\text{COV}(x_1, y_1)$. This is a known quantity in the population; there is a known association between these two variables. What does the model in Figure 2.2, Panel A, imply about this association? The answer can be discovered by substituting the equations of the model for x_1 and y_1,

$$\text{COV}(x_1, y_1) = \text{COV}(x_1, \gamma_{11}x_1 + \zeta_1) \tag{2.7}$$

Covariance algebra aids in rearranging terms so that Equation 2.7 becomes[5]

$$\text{COV}(x_1, y_1) = \text{COV}(x_1, \gamma_{11}x_1) + \text{COV}(x_1, \zeta_1)$$

Exogenous variables are assumed to be uncorrelated with the disturbances, leaving

$$\text{COV}(x_1, y_1) = \gamma_{11}\text{VAR}(x_1)$$

Finally, the variance of the exogenous variable, x_1, is a parameter to be estimated in the model. It appears in the $\mathbf{\Phi}$ matrix. Making this explicit,

$$\text{COV}(x_1, y_1) = \gamma_{11}\phi_{11} \tag{2.8}$$

Equation 2.8 provides the model-implied covariance for $\text{COV}(x_1, y_1)$. In the model in Figure 2.2, Panel A, the covariance between x_1 and y_1 is

[5]The following rules and definitions are used (Bollen, 1989b, p. 21). Defining c as a constant, and x_1, x_2, and x_3 as random variables: (1) $\text{COV}(c, x_1) = 0$, (2) $\text{COV}\ (cx_1, x_2) = c\text{COV}(x_1, x_2)$, and (3) $\text{COV}(x_1 + x_2, x_3) = \text{COV}(x_1, x_3) + \text{COV}(x_2, x_3)$. Note also that $\text{VAR}(x_1) = \text{COV}(x_1, x_1)$.

implied by the model to be due to the variance of $x_1(\phi_{11})$ and its effect on $y_1(\gamma_{11})$.

Of course, COV(x_1, y_1) is only one of the six possible variances and covariances between the three observed variables. The full covariance matrix of the observed variables is

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{VAR}(y_1) & & \\ \mathrm{COV}(y_2, y_1) & \mathrm{VAR}(y_2) & \\ \mathrm{COV}(x_1, y_1) & \mathrm{COV}(x_1, y_2) & \mathrm{VAR}(x_1) \end{bmatrix}$$

For this particular model, there are six variance and covariance elements in the population, and each can be written as a function of the theoretical model appearing in Figure 2.2, Panel A. To provide one more example, consider the covariance of x_1 with y_2. Again, we substitute the equations of the theoretical model to determine what it implies about this covariance.

$$\mathrm{COV}(x_1, y_2) = \mathrm{COV}(x_1, \beta_{21}y_1 + \gamma_{21}x_1 + \zeta_2) \tag{2.9}$$

The variable y_1 is endogenous and requires further substitution:

$$\begin{aligned} \mathrm{COV}(x_1, y_2) &= \mathrm{COV}(x_1, \beta_{21}(\gamma_{11}x_1 + \zeta_1) + \gamma_{21}x_1 + \zeta_2) \\ &= \mathrm{COV}(x_1, \beta_{21}\gamma_{11}x_1) + \mathrm{COV}(x_1, \beta_{21}\zeta_1) \\ &\quad + \mathrm{COV}(x_1, \gamma_{21}x_1) + \mathrm{COV}(x_1, \zeta_2) \\ &= \beta_{21}\gamma_{11}\phi_{11} + \gamma_{21}\phi_{11} \end{aligned} \tag{2.10}$$

In short, the theoretical model implies that the relationship between x_1 and y_2 is due to both the direct effect of x_1 on y_2 and the indirect effect of x_1 on y_2 through y_1. Decomposing covariances in this way shows how the model specified in the path diagram and equations relates to the observed covariances in $\boldsymbol{\Sigma}$. Four implied covariances remain to be calculated to complete the model-implied covariance matrix.

$$\mathrm{COV}(y_1, y_2) = \gamma_{11}{}^2\beta_{21}\phi_{11} + \gamma_{11}\gamma_{21}\phi_{11} + \beta_{21}\psi_{11}$$
$$\mathrm{VAR}(x_1) = \mathrm{COV}(x_1, x_1) = \phi_{11}$$
$$\mathrm{VAR}(y_1) = \gamma_{11}{}^2\phi_{11} + \psi_{11}$$
$$\mathrm{VAR}(y_2) = (\beta_{21}{}^2\gamma_{11}{}^2\phi_{11} + 2\beta_{21}\gamma_{11}\gamma_{21} + \gamma_{21}{}^2)\phi_{11} + \beta_{21}{}^2\psi_{11} + \psi_{22}$$

Pulling everything together, for the three-variable model in Figure 2.2, Panel A, the observed covariance matrix, $\boldsymbol{\Sigma}$, is

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{VAR}(y_1) & & \\ \mathrm{COV}(y_2, y_1) & \mathrm{VAR}(y_2) & \\ \mathrm{COV}(x_1, y_1) & \mathrm{COV}(x_1, y_2) & \mathrm{VAR}(x_1) \end{bmatrix} \tag{2.11}$$

and the covariance matrix that is implied by the model, $\boldsymbol{\Sigma}(\boldsymbol{\theta})$, is

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \begin{bmatrix} \gamma_{11}{}^2\phi_{11} + \psi_{11} & & \\ (\gamma_{11}{}^2\beta_{21} + \gamma_{11}\gamma_{21})\phi_{11} + \beta_{21}\psi_{11} & (\beta_{21}{}^2\gamma_{11}{}^2\phi_{11} + 2\beta_{21}\gamma_{11}\gamma_{21} + \gamma_{21}{}^2)\phi_{11} + \beta_{21}{}^2\psi_{11} + \psi_{22} & \\ \gamma_{11}\phi_{11} & \beta_{21}\gamma_{11}\phi_{11} + \gamma_{21}\phi_{11} & \phi_{11} \end{bmatrix} \tag{2.12}$$

The fundamental statistical hypothesis, $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta})$, means that each element of Equation 2.11 above is equivalent to its counterpart in Equation 2.12. This relationship between $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ is used throughout the rest of the monograph.

To reiterate, the elements of $\boldsymbol{\Sigma}(\boldsymbol{\theta})$, the model-implied covariance matrix, are a function of the researcher's theoretical model as it appears in the path diagram, equations, and matrices. If we were to calculate the implied covariance matrix for the model in Figure 2.2, Panel B, it would be different from Equation 2.12. The hypothesized model changes, so the implied covariance matrix changes as well.

As will be demonstrated throughout the rest of the monograph, the relationship between $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ is critical to all steps in modeling a simultaneous equation model. For example, researchers can sometimes use $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta})$ to solve for the unknown model parameters during identification.

The Implied Covariance Matrix of a Simple Regression

For illustration, we show how to determine the implied covariance matrix of a simple regression model of one exogenous and one endogenous variable:

$$y_1 = \gamma_{11}x_1 + \zeta_1 \tag{2.13}$$

For this model, there are three variances and covariances at the population level, shown in $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{VAR}(y_1) & \\ \mathrm{COV}(x_1, y_1) & \mathrm{VAR}(x_1) \end{bmatrix} \tag{2.14}$$

To determine the model-implied covariance matrix, $\boldsymbol{\Sigma}(\boldsymbol{\theta})$, for a simple regression model, we begin by solving for VAR(x_1), which can be rewritten as COV(x_1, x_1). The variance of the exogenous variable in this model, x_1, as noted earlier, is a parameter to be estimated in the model and appears in the $\boldsymbol{\Phi}$ matrix.

$$\mathrm{COV}(x_1, x_1) = \phi_{11} \tag{2.15}$$

Moving next to COV(x_1, y_1), we substitute for y_1, using Equation 2.13.

$$\begin{aligned} \mathrm{COV}(x_1, y_1) &= \mathrm{COV}(x_1,\ \gamma_{11}x_1 + \zeta_1) \\ &= \gamma_{11}\mathrm{VAR}(x_1) \\ &= \gamma_{11}\phi_{11} \end{aligned} \tag{2.16}$$

Next, we determine the variance of y_1, or COV(y_1, y_1). First, substitute for y_1:

$$\mathrm{COV}(y_1, y_1) = \mathrm{COV}(\gamma_{11}x_1 + \zeta_1,\ \gamma_{11}x_1 + \zeta_1)$$

Using common rules of covariance algebra yields

$$\begin{aligned} \mathrm{COV}(y_1, y_1) = {}& \mathrm{COV}(\gamma_{11}x_1,\ \gamma_{11}x_1) + \mathrm{COV}(\gamma_{11}x_1,\ \zeta_1) + \\ & \mathrm{COV}(\zeta_1,\ \gamma_{11}x_1) + \mathrm{COV}(\zeta_1,\ \zeta_1) \end{aligned}$$

Assuming exogenous variables are uncorrelated with the disturbance leaves

$$\begin{aligned} \mathrm{COV}(y_1, y_1) &= \mathrm{COV}(\gamma_{11}x_1,\ \gamma_{11}x_1) + \mathrm{COV}(\zeta_1,\ \zeta_1) \\ &= \gamma_{11}^2\phi_{11} + \psi_{11} \end{aligned} \tag{2.17}$$

Putting the elements together yields the following:

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta})$$

$$\begin{bmatrix} \mathrm{VAR}(y_1) & \\ \mathrm{COV}(x_1, y_1) & \mathrm{VAR}(x_1) \end{bmatrix} = \begin{bmatrix} \gamma_{11}{}^2\phi_{11} + \psi_{11} & \\ \gamma_{11}\phi_{11} & \phi_{11} \end{bmatrix} \tag{2.18}$$

The fundamental statistical hypothesis, $\mathbf{\Sigma} = \mathbf{\Sigma}(\boldsymbol{\theta})$, means that each element of $\mathbf{\Sigma}$ is equivalent to its counterpart in $\mathbf{\Sigma}(\boldsymbol{\theta})$.

$$\begin{aligned}
\mathrm{VAR}(y_1) &= \gamma_{11}^2\phi_{11} + \psi_{11} \\
\mathrm{COV}(x_1, y_1) &= \gamma_{11}\phi_{11} \\
\mathrm{VAR}(x_1) &= \phi_{11}
\end{aligned}$$

We can solve for the regression coefficient, γ_{11},

$$\begin{aligned}
\gamma_{11} &= \frac{\mathrm{COV}(x_1, y_1)}{\phi_{11}} \\
&= \frac{\mathrm{COV}(x_1, y_1)}{\mathrm{VAR}(x_1)}
\end{aligned}$$

producing the well-known equation available in any introductory econometrics textbook.

Recursive and Nonrecursive Models

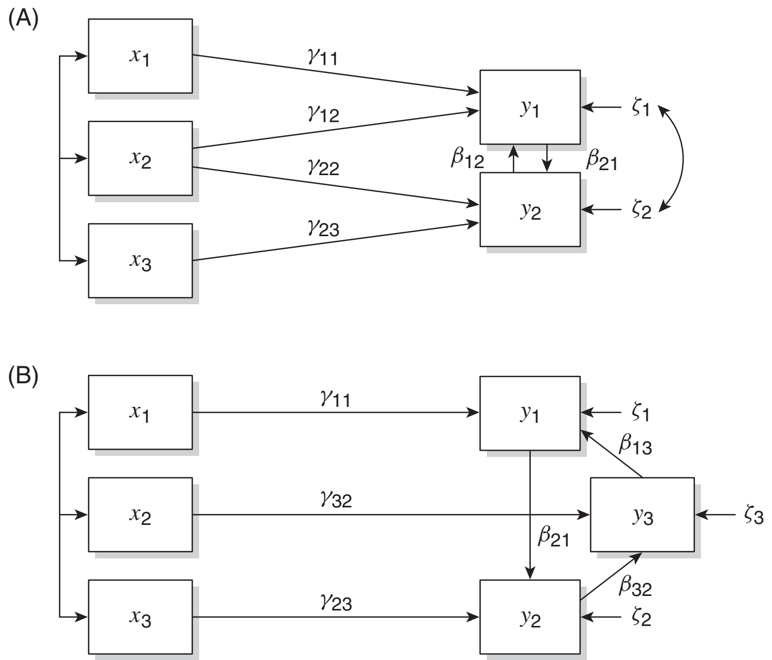
Simultaneous equation models can be divided into two major types: recursive and nonrecursive. A recursive simultaneous equation model has no reciprocal relationships or feedback loops and no covariances among the error terms of the equations (the disturbance of one equation is uncorrelated with the disturbances of all other equations). Formally, in recursive models **B** can be written as lower triangular and $\mathbf{\Psi}$ is diagonal.

A simultaneous equation model is nonrecursive if (1) any of the outcomes in the model directly affect one another (a reciprocal relationship) or there is a feedback loop at some point in the system of equations (a causal path can be traced from one variable back to itself), and/or (2) at least some disturbances are correlated.

In a previous section, we introduced a simple model of two exogenous and two endogenous variables. The model was recursive as demonstrated in both the path diagram and the matrices. Examining the path diagram (Figure 2.1) reveals no reciprocal links or feedback loops. Furthermore, the errors in the equations are not correlated. Examining the matrix equations (2.4 and 2.5) also shows a recursive model: the **B** matrix can be written as lower triangular, and the $\mathbf{\Psi}$ matrix is diagonal.

In contrast, Figure 2.3, Panels A and B show two types of nonrecursive models. Figure 2.3, Panel A, is nonrecursive due to the reciprocal paths between y_1 and y_2 and the correlated error between ζ_1 and ζ_2 (the presence of either would be sufficient to define the model as nonrecursive). Figure 2.3,

Figure 2.3 Two Nonrecursive Models



Panel B, is nonrecursive due to the presence of a feedback loop among y_1, y_2, and y_3. Note that y_1 can be traced back to itself through the change in y_2 and y_3. Similarly, y_2 and y_3 can be traced back to themselves.[6]

Focusing on Figure 2.3, Panel A, the equations of the model are as follows:

$$y_1 = \beta_{12}y_2 + \gamma_{11}x_1 + \gamma_{12}x_2 + \zeta_1 \tag{2.19}$$

$$y_2 = \beta_{21}y_1 + \gamma_{22}x_2 + \gamma_{23}x_3 + \zeta_2 \tag{2.20}$$

[6]A classic theoretical example of a three-variable feedback loop is found in climate science: A warmer climate leads to less snow and ice on the surface of the earth, which leads to less reflection of heat, which makes the climate warmer. Or consider a sleep/stress cycle when sleeping badly leads to more fatigue during the day, which makes an individual less able to cope with stressors and in turn leads to poor sleep.

The matrix equation for the model is as follows:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 \\ 0 & \gamma_{22} & \gamma_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \quad (2.21)$$

and

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$\mathbf{\Psi} = \begin{bmatrix} \psi_{11} & \\ \psi_{12} & \psi_{22} \end{bmatrix}$$

The matrix representation helps clarify why the model is nonrecursive. First, the **B** matrix is not lower triangular and cannot be rearranged to be written as lower triangular. Furthermore, $\mathbf{\Psi}$ is not diagonal; there are off-diagonal elements. Nonrecursive models are more complicated to identify, estimate, and assess than recursive models, as will be covered in later chapters.

Direct, Indirect, and Total Effects

Simultaneous equation models contain direct, indirect, and total effects. Direct effects are effects from one variable to another variable that are not mediated by any other variable in the model. Indirect effects are paths from one variable to another that travel through at least one other variable. Total effects are the sum of direct and indirect effects, representing how much change should occur in the outcome variable for a given shift in the antecedent variable.

Specifying a model as a path diagram allows one to trace paths showing direct, indirect, and total effects. In the path diagram in Figure 2.2, Panel A, for instance, there are three direct effects. Two coefficients show relationships from exogenous variables to endogenous variables, γ_{11} and γ_{21}. One coefficient depicts a path from one endogenous variable to the other endogenous variable, β_{21}. In this model, there is an indirect effect of x_1 on y_2 that works through y_1. The indirect effect is $\gamma_{11}\beta_{21}$. The total effect of x_1 on y_2 summarizes direct and indirect effects, $\gamma_{21} + \gamma_{11}\beta_{21}$. Chapter 6 will describe calculating and testing indirect and total effects in more detail.

Structural Versus Reduced-Form Equations

Thus far, we have focused our attention on the structural equations of the model. Structural equations represent the theoretical model, showing direct relationships between the variables. Structural parameters summarize the direct, "causal" links between variables. Equations 2.19 and 2.20 are examples of the structural equations of a model. Models can also be written in "reduced form," as reduced-form equations.

Reduced-form equations express the endogenous variables solely as a function of the exogenous variables. That is, only exogenous variables appear on the right-hand side (RHS) of the equations. In a model with a reciprocal path, creating the reduced-form equations entails collecting endogenous variables on the left-hand side. In any model, there are the same number of structural and reduced-form equations.

Returning to the nonrecursive model presented in Figure 2.3, Panel A, the structural equations are as follows:

$$y_1 = \beta_{12}y_2 + \gamma_{11}x_1 + \gamma_{12}x_2 + \zeta_1 \tag{2.22}$$

$$y_2 = \beta_{21}y_1 + \gamma_{22}x_2 + \gamma_{23}x_3 + \zeta_2 \tag{2.23}$$

To determine the reduced-form equation for the y_1 equation, substitute for y_2, yielding

$$y_1 = \beta_{12}(\beta_{21}y_1 + \gamma_{22}x_2 + \gamma_{23}x_3 + \zeta_2) + \gamma_{11}x_1 + \gamma_{12}x_2 + \zeta_1$$

Next, multiply, gather, and rearrange terms.

$$y_1 = \frac{1}{1 - \beta_{12}\beta_{21}}(\gamma_{11}x_1 + \beta_{12}\gamma_{22}x_2 + \gamma_{12}x_2 + \beta_{12}\gamma_{23}x_3 + \beta_{12}\zeta_2 + \zeta_1)$$

The reduced-form equation is therefore

$$y_1 = \Pi_{11}x_1 + \Pi_{12}x_2 + \Pi_{13}x_3 + \zeta_1^* \tag{2.24}$$

where

$$\Pi_{11} = \frac{\gamma_{11}}{1 - \beta_{12}\beta_{21}}$$

$$\Pi_{12} = \frac{\beta_{21}\gamma_{22} + \gamma_{12}}{1 - \beta_{12}\beta_{21}}$$

$$\Pi_{13} = \frac{\beta_{12}\gamma_{23}}{1 - \beta_{12}\beta_{21}}$$

$$\zeta_1^* = \zeta_1 + \beta_{12}\zeta_2$$

The y_2 equation is similar:

$$y_2 = \Pi_{21}x_1 + \Pi_{22}x_2 + \Pi_{23}x_3 + \zeta_2^* \qquad (2.25)$$

where

$$\Pi_{21} = \frac{\beta_{21}\gamma_{11}}{1 - \beta_{12}\beta_{21}}$$
$$\Pi_{22} = \frac{\beta_{21}\gamma_{12} + \gamma_{22}}{1 - \beta_{12}\beta_{21}}$$
$$\Pi_{23} = \frac{\gamma_{23}}{1 - \beta_{12}\beta_{21}}$$
$$\zeta_2^* = \beta_{21}\zeta_1 + \zeta_2$$

Reduced-form equations provide information about the total effects of exogenous variables on endogenous variables in a model. In Equation 2.25, Π_{21} is the total effect of x_1 on y_2, which includes the indirect effect of x_1 on y_2 that works through y_1 and includes the reciprocal relationship between y_1 and y_2. Chapter 6 discusses the multiplier, $1/(1 - \beta_{12}\beta_{21})$. The reduced-form equations are central to understanding assessment of simultaneous equation models, as we discuss in Chapter 5.

Instrumental Variables

Identifying and estimating nonrecursive models require understanding instrumental variables. Instrumental variable (IV) estimation was developed for situations where the regressor is correlated with the error term such as occurs in nonrecursive models. In such situations, the regressor is sometimes called "troublesome" or "problematic." Although instrumental variables are frequently treated as a technical solution to an identification or estimation problem—and indeed, this is an important function they serve—we discuss them in this chapter because our view is that they should fundamentally arise from serious theoretical consideration. Furthermore, although instrumental variables are most frequently associated with the literature on limited-information estimators, we emphasize throughout this monograph that careful selection of instrumental variables is necessary regardless of which estimator the researcher employs.

With a problematic regressor, a researcher must find an instrumental variable, which we will call z_1, that is (1) uncorrelated with the disturbance,

$$\mathrm{COV}(z_1, \zeta_1) = 0$$

but (2) correlated with the variable for which it is an instrument,

$$\mathrm{COV}(z_1, x_1) \neq 0$$

In this monograph, we focus on the need for instrumental variables to address a correlation between the error and a regressor due to the regressor's reciprocal relationship with another variable (discussed in Chapter 4).[7] Instrumental variables are also used to correct for a correlation between a regressor and the error due to other causes, including an omitted variable that correlates with the regressor and affects the dependent variable, or measurement error in the regressor.[8]

Example of Instrumental Variables:
Voluntary Associations and Generalized Trust

We demonstrate the use of instrumental variables in a nonrecursive model with an empirical example. This example, from the political science and sociology literatures, will be used throughout the monograph. The theoretical question focuses on the likely interdependent relationship between voluntary association membership and generalized trust. Research on social capital posits the importance of both voluntary associations and trust to the well-being of society (Fukuyama, 1995; Paxton, 2002; Putnam, 1993). But the relationship between the two is likely reciprocal (Brehm & Rahn, 1997; Claibourn & Martin, 2000; Shah, 1998). For instance, scholars have argued that participation in voluntary associations can bring about greater general trust in others due to repeated social interactions, norms of cooperation, and reputation effects (Paxton, 2007). Conversely, those who are more trusting may feel more comfortable interacting with others in an association and therefore may be more likely to join.

We can measure both voluntary associations and trust using data from the 1993 and 1994 waves of the General Social Survey (GSS). In the GSS, respondents report whether they belong to any of 16 types of voluntary organizations, including service, political, youth, and church groups. We computed the number of types of organizations to which a respondent belonged. Generalized trust is measured by three variables: the extent to which the respondent feels people in general (1) are fair, (2) are helpful, and

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[7]In the context of simultaneous equation models, the problematic regressor will typically be indicated by a *y*—as an endogenous variable in the system of equations.

[8]Both the well-known omitted variable problem and measurement error are issues that confront all analytic techniques using observational data, including ordinary least squares, logistic regression, and so on, and are not issues unique to simultaneous equations models.

(3) can be trusted.[9] For our simple example, we created a single factor score estimate that is the combination of the three indicators of trust.[10]

To keep the example simple, we include one predictor that influences both endogenous variables: the level of education of the respondent in years. Theory and prior research suggest that more educated respondents should belong to more voluntary associations and also have greater generalized trust. To identify and estimate this model, we also need at least one instrumental variable for each endogenous variable. This must be a variable that does not have a direct relationship with the outcome variable in a given equation, or any omitted variables that influence the outcome variable. In a path diagram, an instrumental variable will appear as a variable predicting one endogenous variable but not the other.

For voluntary association membership, one such possible instrumental variable is the presence of young children less than 6 years of age. Although there is little reason to expect that the presence of young children will affect one's trust in others, they likely influence the amount of time available for association participation. For trust, a possible instrument is a measure of whether the respondent has been burglarized in the past year. Although there seems little reason for such an event to affect one's memberships, experiencing a burglary should affect one's trust in others.

Figure 2.4 displays this model in a path diagram. The variables y_1, voluntary association memberships, and y_2, generalized trust, are in a reciprocal relationship; x_1, children less than 6 years of age, influences voluntary association memberships but not trust and therefore serves as an instrumental variable in the model; x_2, education, influences both voluntary associations and generalized trust; and x_3, burglary, is hypothesized to influence generalized trust but not membership in voluntary associations. The variable x_3 therefore serves as an instrument for generalized trust in the model displayed in Figure 2.4. The reasoning provided for the instrumental variables in this model is at this point entirely theoretical. Tests described in Chapter 5 can help a researcher determine whether an instrumental variable is valid.

It is also helpful to have an overidentified model. To overidentify the model, we modify the exactly identified model by adding a second excluded instrumental variable to each equation (see Figure 2.5). In the

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[9]There are three questions regarding trust, helpfulness, and fairness: "Generally speaking, would you say that most people can be trusted or that you can't be too careful in life?" "Would you say that most of the time people try to be helpful, or that they are mostly just looking out for themselves?" "Do you think most people would try to take advantage of you if they got a chance, or would they try to be fair?"

[10]One could instead model trust as a latent variable with three indicators (Paxton, 1999), which would make the resulting example a general structural equation model (Bollen, 1989b).

Figure 2.4 Path Model of Voluntary Associations and Generalized Trust

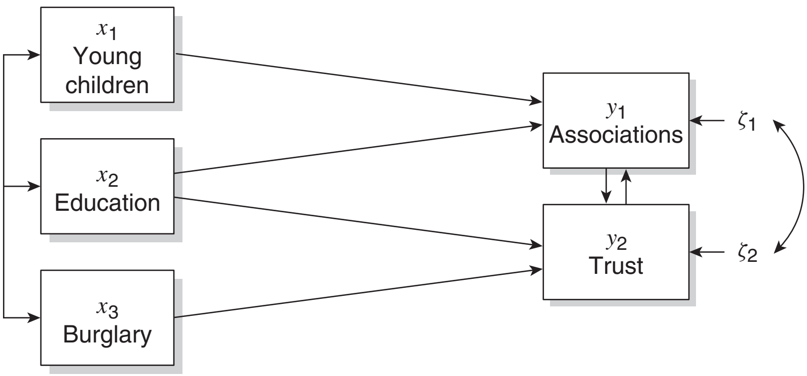
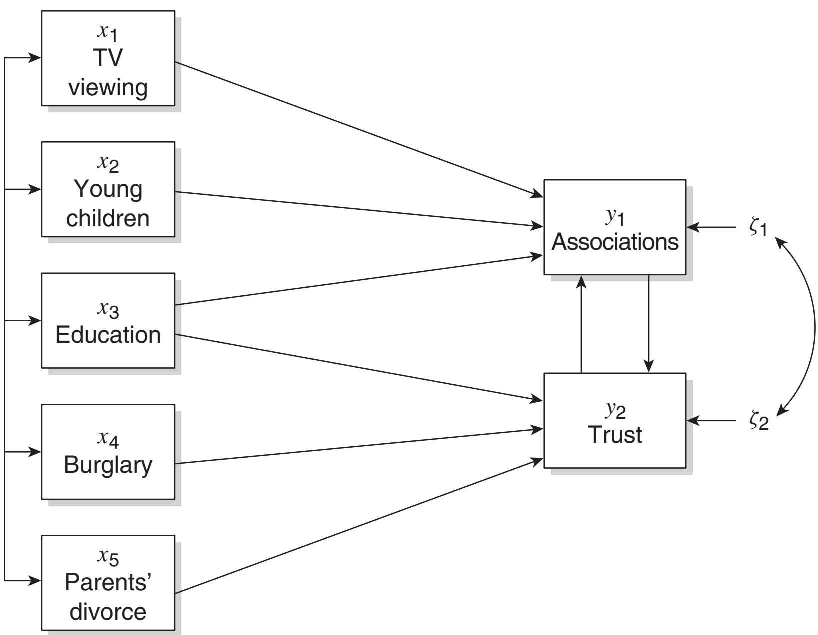


Figure 2.5 Overidentified Model of Voluntary Associations and Generalized Trust



overidentified model, hours of TV viewing influences association participation, and having experienced the divorce of one's parents influences trust.

Examples of Instrumental Variables in Published Research

Finding appropriate instrumental variables is a challenging task and should not be taken lightly. But although challenging, the task is not impossible. In this section, we provide additional examples of instrumental variables used to identify and estimate models in published research.

- Kritzer (1984) used survey data from husbands and wives to test the likely reciprocal effect of political party identification between spouses. That is, husbands' political party identification may influence wives' party identification, and vice versa. Kritzer used the party identification of parents as instruments. His logic was that although we might expect the party identification of a spouse's, say the wife's, parents to affect her party identification, there is little reason to suppose that the wife's parents' party identification would affect her husband's party identification (after accounting for her own party identification). Parents' party identification acts as an instrumental variable in this model.
- Levitt (1996) wanted to investigate the effect of prison incarceration on crime rates. Certainly, increasing the prison population could lead to decreased crime rates. But increased crime rates could also increase incarceration, creating a reciprocal relationship and a problematic regressor. Levitt used litigation related to prison overcrowding as an instrument for prison incarceration. He argued that prison litigation could cause prison populations to decrease but would not directly affect crime rates. In a similar manner, Hoxby (1996) used the passage of laws allowing union activity as an instrument for teachers' unionization in its effect on student outcomes.
- Some studies use attitudinal measures as instruments. A study of the reciprocal relationship between females' labor force participation and fertility expectations used a measure of attitude toward the workplace to predict labor force participation (assuming that it does not directly affect a respondent's fertility expectations) (Waite & Stolzenberg, 1976). This same study also posited that one's ideal family size will affect fertility expectations but will have no direct effect on actual labor force participation.

- Sadler and Woody (2003) theorize that individuals in interaction are influenced by their interaction partner's level of dominance (a reciprocal relationship between two individuals). To create instruments, Sadler and Woody rely on other theories suggesting that individuals bring their own interpersonal style to an interaction as well. Thus, both individuals will bring their own "trait dominances" to the interaction, and trait dominance acts as an instrumental variable in the model. That is, an individual's trait dominance will influence his or her own interactional dominance. But it should not influence his or her interaction partner's interactional dominance directly.
- A study estimating the reciprocal relationship between crime and residential mobility posited population size as an instrument for crime (Liska & Bellair, 1995). Although population size may influence crime, there is little reason to suppose it affects residential mobility. The authors also hypothesized that government revenue per capita—as a proxy for tax burden—may affect residential mobility without affecting crime (making tax burden an instrument for residential mobility). Note that this latter specification assumes that social services do not affect the level of crime, which may or may not be theoretically justified. Statistical tests described in Chapter 5 can help a researcher further evaluate potential instrumental variables.
- Barro and McCleary (2003) investigate whether the religiosity of a population influences a country's economic growth. But there could be a return effect since economic development may cause individuals to become less religious (the secularization hypothesis). As one instrument for religion, Barro and McCleary use the establishment of a state religion. Establishment of a state religion should arguably influence religiosity in a country, but, as state religions were often declared centuries ago, not economic growth in the present period. See Young (2009) for an outstanding discussion and assessment of state religion as an instrument for religiosity.
- Ansolabehere and Jones (2010) ask whether a constituent's perceived agreement with their congressperson on policy issues increases their approval of that congressperson. But it might be that constituents who approve of their congressperson tend to assume they agree on questions of policy. Ansolabehere and Jones had information on *actual* roll call voting by congresspeople and could use that as an instrument for constituents' perceived roll call voting. Reasonably, actual roll call votes can only influence constituent approval through constituent perceptions of those votes.

These are just a few examples of the variables researchers have used as instrumental variables when modeling nonrecursive models. Careful theory and some creativity can produce useful instrumental variables. Indeed, solid reasoning and clever research designs can take precedence over many of the statistical features of these models. In research using instrumental variables to address problematic regressors of a variety of types, researchers have used physical features as instrumental variables (e.g., the number of rivers in a metropolitan area as an exogenous source of the number of political units and subsequent segregation; Cutler & Glaeser, 1997); a lagged version of the variable of interest (assuming that there are no additional effects from earlier time points; Markowitz, Bellair, Liska, & Liu, 2001); and "simulated instruments" (Hoxby, 2001). For other examples, see Murray (2006b, chap. 13).