WHAT YOUR COLLEAGUES ARE SAYING . . .

"Through real classroom interactions, deep mathematical analysis, and thoughtfully constructed problem strings, Pam Harris shows us how additive reasoning, multiplicative reasoning, and proportional reasoning are connected and figure-out-able. This is a must-read book for anyone who is striving to teach 6-8 math in a way that makes it make sense."

Peter Liljedahl
Professor of Math Education, Simon Fraser University
Author, *Building Thinking Classrooms*
Burnaby, BC, Canada

"Pam Harris makes the often-messy work of 'reasoning over rules' feel concrete and teachable—showing middle grade teachers how to move students beyond algorithm-following into real mathematical thinking. With clear learning progressions and practical routines like Problem Strings that help students see the math and make sense of relationships, this book is both a mindset shift and a ready-to-use roadmap."

John SanGiovanni
Instructional Facilitator for Elementary Mathematics
Howard County Public Schools
Ellicott City, MD

"Pam Harris has written the book teachers have long needed. Fractions and proportional reasoning are the bedrock of algebraic thinking, and she makes both accessible with practical, research-grounded strategies. This book doesn't just build understanding—it helps both students and teachers reclaim their confidence."

Pamela Seda
Co-author, *Choosing to See: A Framework for Equity in the Math Classroom*
Atlanta, GA

"Pam Harris has done it again! This book is a masterclass in understanding and teaching middle school math concepts. Whether you are new to teaching middle school math and looking for a foundation to build from or a veteran looking to deepen your understanding, *Developing Mathematical Reasoning* is a must-have resource. The demonstrative visuals and sample dialogues provide a rich learning experience for the brains of all teachers."

Liesl McConchie
Math With the Brain in Mind
Author, *Building a Positive Math Identity*
San Diego, CA

"*Developing Mathematical Reasoning: The Strategies, Models, and Lessons to Teach the Big Ideas in Grades 6–8* is an energizing read that helps middle school teachers move students beyond procedures to making sense of ratios, proportional reasoning, and operations through purposeful tasks. The book feels immediately usable for Grades 6–8 classrooms, offering practical moves that strengthen conceptual understanding while building coherence across the middle school math curriculum."

Carrie S. Cutler
Clinical Associate Professor, University of Houston
Houston, TX

"The thing I'm most grateful for in this book is that on every page you can feel the heartbeat of a practitioner. While Harris has deep knowledge of educational research underneath everything she does, *Developing Mathematical Reasoning* is brimming with classroom-ready insights that will equip and enrich every teacher who picks it up and puts it into action."

Eddie Woo
Professor of Practice in Mathematics Education,
University of Sydney
Sydney, New South Wales, Australia

"This is a powerful and practical guide for any teacher who wants their students to truly understand mathematics, not just perform it. Pam Harris, once again, challenges the idea that math is about memorizing steps and instead shows how students can build strategies through reasoning and relationships. The strength of this book is that it doesn't just tell you what to believe; it shows you how to teach differently. The Problem Strings, the focus on making thinking visible, and the emphasis on multiplicative reasoning make this an incredibly valuable resource for the classroom. It's the kind of professional learning that genuinely changes practice."

Chris Hogbin
Math Teacher and Founder, Number Hive
Canberra, Australian Capital Territory, Australia

"Pam Harris and her team have done it again! This book brings concepts to life through engaging vignettes, real classroom examples, and visuals to help us make sense of the math. Grounded in sense-making, it invites deep thinking while remaining enjoyable to read. It's an accessible, thoughtful resource that reflects her belief that *math is figure-out-able*. It also helps teachers connect middle school ideas to elementary models, strengthening coherence across grades."

Marria Carrington
Director, Mathematics Leadership Programs
Mount Holyoke College
South Hadley, MA

"Pam Harris's use of Problem Strings transforms mathematics instruction from a linear sequence of isolated, bite-sized lessons into a series of conceptual landmarks, allowing students to navigate their own pathways toward understanding. For educators seeking to foster deeper mathematical thinking and to support students in modeling and articulating their reasoning, this book is an invaluable resource."

Leslie Mohlman
Secondary Mathematics Teacher, Alpine School District
Lehi, UT

"This book is a must-read for teachers who are not only looking to build their content knowledge around traditionally difficult middle school content but also give actionable steps to fill the gaps in learning for students to be successful."

Lauren Gangel
5th/6th Grade Math Teacher, Twin Cedars Community School District
Hamilton, IA

Developing MATHEMATICAL REASONING

The Strategies, Models, and Lessons to Teach the Big Ideas in Grades 6–8

PAMELA WEBER HARRIS

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Contents

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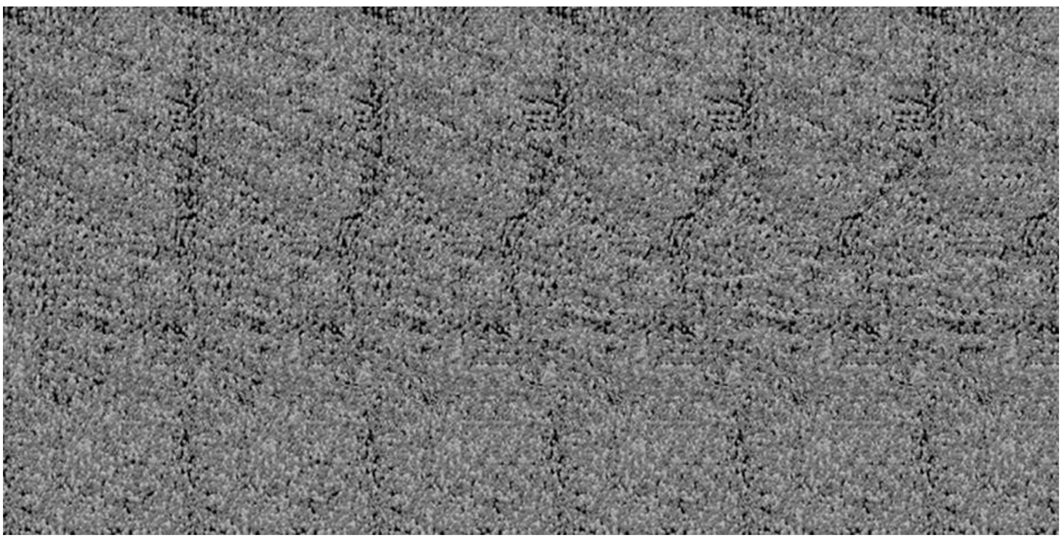
Videos and a book study companion guide may also be accessed at **http://www.mathisfigureoutable.com/dmr/6-8**

Preface

Trying to learn real math in a fake math classroom is a lot like looking at an autostereogram.

Autostereograms are pictures made of colored dots that at first glance look like visual noise. If you have the skill to focus your eyes just right, that visual noise resolves into a three-dimensional image that will appear to float in front of the page (Figure 0.1).

FIGURE 0.1 ● Autosterogram or Magic Eye



Source: Adapted from https://en.wikipedia.org/wiki/File:Stereogram_Tut_Random_Dot_Shark.png with CC Attribution-Share Alike 3.0. Retrieved October 19, 2024.

In this example, focusing beyond the page about 3 inches will reveal the image of a shark. Some of you will see the shark immediately, but many of you will not. Some will struggle and eventually see the shark. But some of you will probably never see the shark.

Think of this autostereogram of a shark as learning math (Figure 0.2).

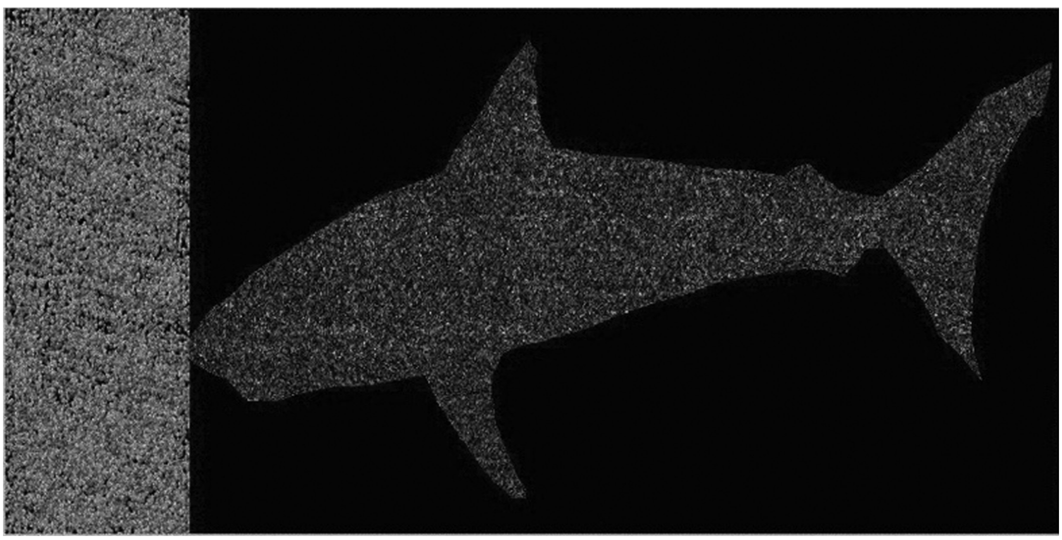
Some of you will say, “Weird. The teacher is calling this a shark, but it doesn’t look like what I thought a shark is. They say to memorize these weird squiggles and bits of color, so that must be what a shark is—I guess I will memorize these five squiggles and this horizontal wave—that must be what a shark is. I'll memorize that so that when I see it later, I'll know it's a shark." This was me.

Others, without guidance from a teacher, have the natural inclination to very quickly focus their eyes to see the shark. "There is a shark, surely everyone can see the shark. It's obviously right there. I wonder why the teacher is talking about five squiggles and the horizontal wave. Weird. It's just a shark." This was my son.

Still others can't see the shark but aren't willing or able to play the game the first group does. The nonsensical memorization of lines and dots when there is supposed to be a shark isn't enough for them to hold on to. Math becomes increasingly stressful. Students become convinced they just "aren't math people."

Who can blame them? Not me, not when we tell them there is a shark in the water, and then when they say they can't see it, we just throw more and more confetti dots at them.

FIGURE 0.2 ● Part of the 3-D Image Seen When Viewed Correctly



Source: Adapted from https://en.wikipedia.org/wiki/File:Stereogram_Tut_Random_Dot_Shark.png with CC Attribution-Share Alike 3.0.

A lot of really good work in the mathematics education field today focuses on getting students to be willing to try to focus on the shark again. *Building Thinking Classrooms* by Peter Liljedahl (2021) falls into this category. After years of nonsense, many students aren't willing to try to make sense of anything.

Disruptions to the classroom norm like vertical-nonpermanent surfaces and randomly chosen groups (as Liljedahl recommends) can do wonders. The willingness to try squinting (focusing) again is crucial.

But a willingness to try is not enough. All students need guided, deliberate instruction aimed at how to see the shark. They need focused, purposeful instruction to do the mental actions that *math*-ers do. That's where this book comes in.

When I say math is figure-out-able, I'm saying everyone can be taught to see the shark. Everyone can be taught to see real math, to *math* their world.

TIP

"*Math*-er" (a term coined by Peart Crayton, 2026 to refer to anyone who does math) for the purposes of this book describes someone who does the mental actions of logically reasoning using mathematical relationships.

When I say math is figure-out-able, I'm saying everyone can be taught to see the shark.

Picture a large room full of teachers eager (or not so eager) to learn more about teaching students to reason proportionally, to develop the reasoning associated with fractions and proportions. Picture me facilitating the workshop, needing to set the stage for just what we'd be getting into over the next few days. Picture me bringing up Susan J. Lamon's book *Teaching Fractions and Ratios for Understanding* (1999), where she estimates that 90% of adults do not reason proportionally.

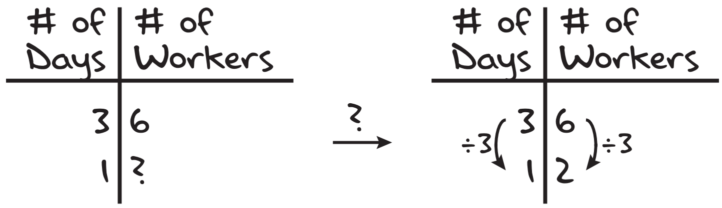
Every teacher in this room is concerned with teaching proportional reasoning. For some, this only confirms what they've known or suspected for years, that despite all their best efforts, students just aren't grasping the content they teach. Others scoff, confident anyone leaving eighth grade without proportional reasoning must not have been drilled hard enough on the appropriate algorithms. Others dismiss the statistic out of hand. The situation can't really be that bad.

This is not my first rodeo, so I anticipated these reactions. A statistic may or may not make an impact on a person, but I've continuously found that experiences do. My next move is to distribute Lamon's ten questions that, if reasoned through, demonstrate that a person is reasoning proportionally.

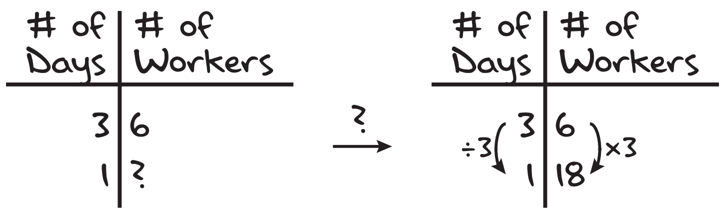
The following is one of the questions:

> Six workers can build a house in 3 days. Assuming that all of the workers work at the same rate, how many workers would it take to build the house in 1 day?

Many in the room confidently reason that what you do to one number, you do to the other. They see 3 days divided by 1 day (divide by 3) in the question, so they divide 6 workers by 3, thus the answer is that 2 workers take 1 day to build the house.



Others reason that if you divide the time it takes to build the house by 3, it would take 3 times the number of workers, 18 workers to build the house in 1 day.



Which reasoning makes more sense to you?

What are the implications of half a room of middle school teachers reasoning incorrectly?

Lively discussions ensue about who is correct. I let the uncertainty sit, knowing that we would come back to it the next day after all the learning.

Could it be that many of us have been trying to teach math we don't understand ourselves (or at least could understand much better), with many of the rest not knowing how to teach the understanding we do have?

I don't think this is news for most of us, though this may be the first time you've seen it said out loud (written out?). It isn't news that teaching fractions and proportions are a mess of stress and misunderstanding. A mess we can track back to our own stressed and opaque experience "learning" fractions and proportions when we were students.

The good news is that it doesn't have to be that way.

Back to my story. We dive in and use Rich Tasks and Problem Strings to develop proportional reasoning with these wonderful teachers. Then, at the end of the next day, everyone confidently reasons through similar house-building-worker problems. Many teachers are hopeful, now armed with ways to help their students not fall into the same trap of "stepping" through problems instead of reasoning through them. Now they understand why the right answer is correct and have a solid foundation for teaching that same understanding to their students.

All of their students. Because all students can do more real *math*-ing than fake *math*-ing.

Proportional Reasoning doesn't have to be this weird thing only 10% of the population can do. It can be accessible and joyful to teach.

Proportional Reasoning is figure-out-able.

In my early years of teaching, I started to realize that what I thought was math was only a shadow of what math really is.

The inkling that I was missing something massive had started when my eldest son—in elementary school—started exploring math in ways I never had, even as a high school mathematics teacher. This drove me to dive into every bit of existing research I could get my hands on.

I began to realize that the math I had learned and the math I taught my high school students were more akin to sequencing memorized bits of information than real mathematical reasoning. We memorized when to use which procedure and lists of steps, and we memorized the answers to pieces of those steps. Undoubtedly, some of my students were like my son, managing to piece together the logic and patterns behind those steps and really reason mathematically, but most of us, myself included, did not.

Some of you reading this are surprised that a high school mathematics teacher was not actually *math*-ing. Others are not surprised but are very frustrated. You know math makes sense, and has an underlying logic, but you don't understand why your teachers refused to teach you that logic. You might be wondering if the reason your teachers didn't teach you the underlying logic is because they didn't know that logic exists. Still others are like me, who bought into the myth that to do math was to rote-memorize and mimic. Students who tried to hold on to dozens of algorithms and formulas and, without really understanding why many of them work, tried to pick the right one to get the right answers.

If you are in this last group, you might also be starting to get an inkling that math isn't quite what you thought it was. This can be very unsettling. This awareness gave me something of an early midlife crisis. Maybe you aren't in any of those groups. Maybe you know that math makes sense, that it is figure-out-able. That math taught as figure-out-able engages and enlightens. What you may not know is how to teach it that way.

If, as a student you rocked at memorizing squiggles, then as a teacher you probably rock at teaching memorization of squiggles with rhymes, games, and drills. Your students have gotten the best possible start you could have given them. But most, like yourself, have not yet seen the actual shark in the water. You *want* to see the shark and help your students *math*. And now that you'll know better, you can do better.

If as a student you saw the shark powerful and attention demanding, then as a teacher you probably started out confused about why most of your students didn't see what you saw. Maybe you found ways to help with that, maybe you didn't. Most likely driven by good intentions and frustration, you ended up teaching the same way you had been taught and accepting that only a rare few students would understand math the way you do. You want to help students see the beautiful sharks and are hungry for effective ways to do that.

If as a student you had the looming, insistent feeling that there was something else in the water with you, and were occasionally freaked out that your teacher refused to acknowledge the fin cutting through the surf or explain the bite marks in your surfboard, then as a teacher you know something is up. If you knew the story about math your teachers were trying to sell you didn't add up, and were frustrated by their refusal to acknowledge that, then you want to improve. So as a teacher you do your best, sharing what you have been able to piece together. And you wish you could do more.

Whatever group you may fall into, this book will help you teach more real math to more students.

Even 25 years ago there was much research on this subject, but finding the good stuff and figuring out how to piece it together took my time, active investigation, experimentation, and my background in higher math to sort the usable from the less useful.

What I've found is that the single most powerful way to get all students—regardless of their current experience level—to see the shark of real mathematics, is to provide an experience where the shark jumps right out of the water straight at them. In practice this looks like carefully crafted Rich Tasks and teaching routines like Problem Strings. This is particularly true for disenfranchised students who may have long since given up on ever seeing the shark.

Real *math*-ing consists of reasoning using connections, understandings, and relationships. Fake *math*-ing is memorizing disconnected sets of facts and mimicking procedures, where each adds yet another ball of confusion to be juggled on top of the last one.

Real math-*ing consists of using connections, understandings, and relationships.*

This book is the guide to getting sixth-, seventh-, and eighth-grade students shark seeing—doing the mental actions that mean you are *math*-ing. It is the result of the research and experimentation I have done in the last 25 years to learn how to build students' brains to do more math, rather than merely burden those brains with more disconnected sequences of steps.

ABOUT THIS BOOK

Chapter 1 summarizes the contents of the first book in this series, *Developing Mathematical Reasoning: Avoiding the Trap of Algorithms* (Harris, 2025). While it is intended for that book to be read before this one, this chapter serves as an abbreviated launch point or refresher. In short, algorithms have numerous shortcomings when used as teaching tools. Chapters 2 and 3 work through the foundation of additive and multiplicative reasoning students would have ideally built in Grades 1–5. It isn't news to you that some or most of your incoming students might not have learned all the material previous grades are meant to cover. These chapters are here to give you the foundation you need to catch your students up while still developing your grade-specific material. Chapter 4 defines Proportional Reasoning, focusing on the interconnected sources of meaning for rational numbers and how to teach them, and the connection to functional reasoning with $y = kx$. Chapter 5 dives into Proportional Reasoning strategies and how to help students develop them. Chapter 6 describes the classroom tasks to develop all of the above. Chapter 7 delineates the key differences between models and strategies, illuminating the most important models to make thinking visible and to use as tools for reasoning. Chapter 8 lays out the next best steps you can take in your classroom today, tomorrow, and moving into the rest of the school year.

This is the third grade-specific book Corwin and I have published on a sixth-month cadence (K–2, 3–5, and now 6–8); once this book is released, we will publish one more (9–12) that will offer more ideas, more practice, and more practical advice. These books will be complementary to the anchor volume and this book.

Neither this book nor the rest of the (currently!) planned series have a focus on geometry, data, or measurement. Due to time and space constraints, I choose to restrict this series to the fundamentals of mathematical reasoning, as they are foundational to meaningful geometry, data, and measurement.

LANGUAGE USE IN THIS BOOK

There are a few terms that will be helpful to parse out before continuing.

- **Mathematical Reasoning**

  As used in this book, the term *mathematical reasoning* does not mean just a general ability to think. This is not a fuzzy "think better" approach that doesn't include doing the math and getting results. Mathematical reasoning is about building stronger brains and expects more, not less, from students, giving them the tools to be successful at *math*-ing. It demands increasing sophistication of strategy. We meet students where they are and then develop from there. For example, students will not only know their multiplication facts, they will actually own them and be able to use the relationships in problems. It includes content-specific milestones such as understanding of place-value, the four operations, fractions, ratios, and so forth.

- **Sophistication**

  Sophistication, as used in this book, is a descriptor of how different levels of mathematical reasoning relate to each other. This includes how the domains of reasoning relate to each other, such as additive reasoning being more sophisticated than counting, and also how individual strategies within a domain relate to each other. For example, both the Smart Partial Products Over and the Flexible Factoring major strategies use multiplicative reasoning. However, the latter requires more simultaneity and developed understanding, and is, therefore, more sophisticated.

Sophistication is always a descriptor of a thought process, never a descriptor of a thinker.

- **Problem Strings**

  Problem Strings are everywhere in this book, because Problem Strings are the single best teaching routine for building sense making and teaching the major strategies. Problem Strings are deliberately ordered sequences of problems designed to develop an important big idea, model, or strategy. They are always meant to be teacher-led routines, with the teacher facilitating learning by revealing each problem one at a time, drawing out—modeling—student thinking, and crafting class discussions after each problem to highlight the patterns at work for the given strategy.

  For more examples of Problem Strings for middle school, see *Building Powerful Numeracy for Middle & High School Students* (2011) and *Lessons & Activities for Building Powerful Numeracy* (2014). Problem Strings are also discussed in greater detail in Chapter 7.

- **Strategy and Model**

  The term *strategy* can mean instructional strategy or general problem-solving strategy, but in this book, strategy means how you use mathematical relationships to reason through a problem. This is different from how you represent a strategy—that is a model.

  The term *model* is commonly used with many different definitions in mathematics teaching. In this book, model means "representing student thinking," which relates to the "representation of student thinking" and also as a "tool for thinking."

  For more on the differences between strategy and model, see Chapter 8. The full breakdown is reserved for the end of the book, as the context given by the proceeding chapters will aid greatly in understanding it.

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A hearty thanks to my son, Cameron Harris, for helping me find real *math*-ing and writing this book.

Thank you to Kourtney Lambert Peters for helping write this book—your experience in the classroom and attention to detail is greatly appreciated. And your dogged poking to keep me on track is helpful! You provided great video footage with very real middle school students! Thanks for the phrase, “formulas don’t initiate a to-do sequence.” And yes, to you and Brandon Pelter, for thinking carefully about variable parts. :)

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Thank you to Sue and Kira and the rest of the gang at Math Is Figure-Out-Able for keeping the ship sailing while my head is down writing or I’m traveling. To Susan Smith, glad to have you on board—thanks for jumping in! And thank you to Ann Latham for your meticulous work with permissions.

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Many thanks to the teachers at Chapa and Wallace middle schools for inviting me into your classrooms to film. You and your students rock!

With great respect and gratitude, I acknowledge the thousands of teachers and their students who have let me in their classrooms, joined our coaching group, and taken our workshops—online and in person—and learned along with me. Your dedication to your craft is admirable. I hope you find this book helpful in your continued journey.

Thank you dear reader for your commitment to your craft. You’re not paid enough nor receive enough respect, but you’ve dedicated your life to lifting where you stand and I honor you for it.

And to the author of life, thank you Lord for giving me a message worth sharing and the people around me to help me get it to the world.

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Leslie Mohlman
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Lehi, UT

About the Author

Pamela Weber Harris is changing the way we view and teach mathematics. Pam is the author of several books, including the *Numeracy Problem Strings K–5* series, *Building Powerful Numeracy*, and the *Foundations for Strategies* series. As a mom, a former high school math teacher, a university lecturer, and an author, she believes everyone can do more math when it is based in reasoning rather than rote-memorizing or mimicking. Pam has created online *Building Powerful Mathematics* workshops and presents frequently at national and international conferences. Her particular interests include teaching real math, building powerful numeracy, sequencing rich tasks to construct mathematics, using technology appropriately, and facilitating smart assessment and vertical connectivity in curricula in schools PK–12. Pam helps leaders and teachers make the shift that supports students to learn real math because math is figure-out-able!

PART I

Setting the Stage

Chapter 1: Mathematics for Teaching

CHAPTER 1

Mathematics for Teaching

After years honing the routines and classroom direction needed to teach elementary students how to see the shark of real mathematics (see Preface), I find myself facilitating a workshop with middle school teachers in a large room just off the school's library. I'm in a room with very large windows of the near-constant-distraction variety.

It is going great. In short order, we build enough multiplicative reasoning that the teachers are reasoning through division problems like pros. More importantly, they see how this approach would help all their students have more access to more math.

Then one of the teachers asks an important question, one that gets asked all the time. Heather says, "I get how this works for whole numbers, but what about decimal division?"

At that point, I know instinctively that decimals will be just as figure-out-able as whole numbers. If it isn't about social convention, what we call things, all math can be figured out. That's what makes it math.

So we dive in. And it is messy. It is unguided. It feels a lot like I had dropped the teachers onto an island to rediscover math by themselves.

It's not hard to see why. I hadn't done the work to figure it out myself yet, let alone the work to figure out what routines and teaching strategies we would need to deliberately guide students (and teachers!) through building their own reasoning. Knowing that division of decimals is figure-out-able is not enough. Seeing the shark of math is not enough. We have to know how to teach others to see the shark. We have to know how to create experiences for students, to guide them along the way, and what we as teachers need to be able to guide them, without stripping math down to the "fossilized remains" that algorithm-centered teaching necessarily does (Freudenthal, 1973).

What I really could have used in that moment was a book written by someone who had done the work and research.

I really could have used *this* book.

It's been 19 years since that day in a workshop off a library with too-large windows, and I've done the work.

WHAT'S THE PURPOSE OF LEARNING MATH?

For a moment, leave the world of education as it currently exists. If you had to argue for the inclusion of mathematics in K–12 education in any form, what arguments would you make?

I argue that mathematics and literacy are the two most powerful tools a person can leverage for understanding, organizing, and making an impact on the world. From there, a number of consequences logically flow. First, nothing, no matter how much it uses the trappings of mathematics, is worth spending valuable classroom time on if it doesn't aid in a student's ability to understand, organize, and make an impact on their world. Second, among possible ways to spend limited classroom time, priority must be given to those activities and approaches that make the greatest impact.

I argue that mathematics and literacy are the two most powerful tools a person can leverage for understanding, organizing, and making an impact on the world.

Which means, third, that in the best-case scenario, algorithms and step-by-step procedures as answer-getting tools are near the very bottom of the barrel of ways to spend classroom time. In the worst case, teaching to mimic the steps of algorithms can actively inhibit many students' ability to understand, organize, and make an impact.

Let's return to the opening hypothetical to illustrate why. If your argument for the inclusion of mathematics in K–12 education is that students need to be able to get the answers to math problems as fast and easily as possible, math education ceases to need to be a pillar of the K–12 experience and can instead be reduced to a few days covering how to use a calculator, or now in the age of AI, a few minutes writing prompts.

Rote-Memorize: to memorize isolated facts without the context that give them meaning. This could include memorizing a table of multiplication facts the same way one would memorize a list of capital cities.

FREQUENTLY ASKED QUESTIONS

Q: But Pam, that's what math is. The definition of math is to rote-memorize and mimic steps.

A: If you're thinking that, it's understandable. I invite you to consider that you might have been trapped by algorithms. Take heart! This book will help free you.

Q: But Pam, students must learn the steps of an algorithm because that is how they learn the concepts behind it all.

A: If you think that you learned to *math* (Peart Crayton, 2026) because your teachers had you repeat steps of algorithms, I invite you to consider that you actually did much of your learning on your own. Somehow you saw through the steps and made connections. Most people don't do that on their own with such a low dose of the patterns behind the algorithms. The good news is that with a high enough dose, everyone can.

Let's dive in and experience what it can look like to help students see the shark of mathematics, *math*-ing: doing the mental actions of a mathematician and finding success.

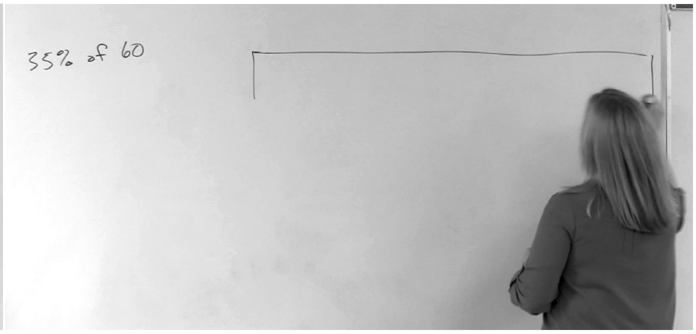
TIP

Read this guided walk-through for the most important moments in the Problem String. At the end, there will be a QR code to watch the unabridged video.

Join me with an eighth-grade class near Austin, Texas where I facilitate a Problem String adapted from my Lessons and Activities for Building Powerful Numeracy (2014). These students participated in a couple of Problem Strings with me the day before.

I begin, "So, yesterday, we did some work with percents. Let's show off a little bit how you guys are thinking about percents. What is 35% of 60?"

I repeat the problem as I write it on the board and draw a really long percent bar. "Remember, yesterday? What were we doing? What did that bar represent? What did that mean?"

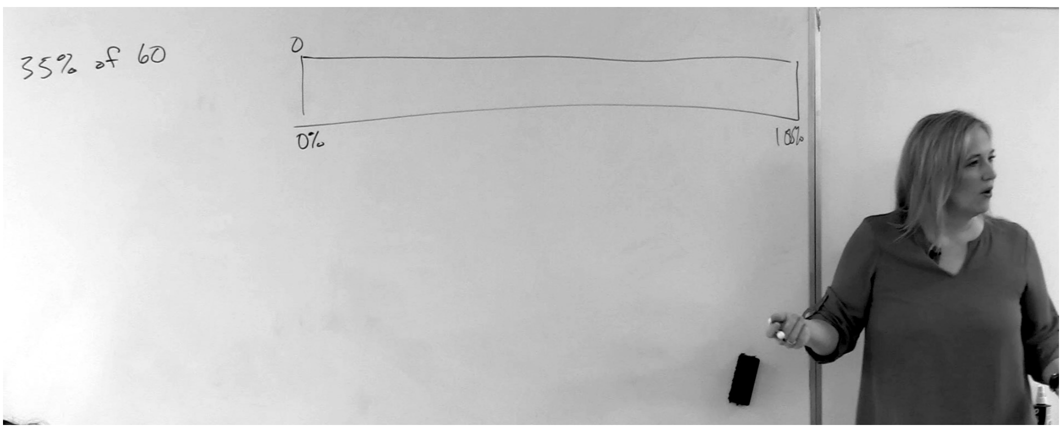



TIP

Draw the percentage bar really long so that you can fit all of the values as proportionally as possible.

With student input, we begin to label what we know. Students remember that yesterday we had talked about download bars that show how much of a file of music students had downloaded.

I ask, “And then in this case, do we know what the total was?” I wish I would have stayed in context by asking, “Do we know what the total *terabytes* was?” There are about to be a lot of numbers thrown around. Keeping everything in context gives students a higher possibility of making sense of what’s happening.



When Isaiah says, “60,” I repeat, “You’re like, ‘It’s 60.’ We’re trying to download 60,” and I write 60 above the 100%. “But right now, we are only at 35% of 60. So, I’m curious. How many terabytes have you downloaded if you’re only at 35% of the 60? Like, where would it stop if it were at 35%?” I motion along the bar, back and forth with a questioning look.

Students call out, “More toward the end.” “The left.” “Like a little bit before 50.”

I repeat, “A little before 50. Do you guys agree? 35% is a little before 50%?”

As students nod, I continue, “Could you think of some percents that you could find that you could mark on here? Go ahead and work on that. I heard a 50 over here. I wonder if 50 would be helpful? Could you find some percents that would be helpful to maybe add up? Remember, yesterday we added up a bunch of percents to get to the one we were looking for. Today, you guys get to decide on a happy percent. Go.”

As I circulate, conferring with students, I ask, "What kind of chunks would you need to get to 35%?"

Savannah suggests, "25."

I respond, "25% sounds really helpful. Could you find 25% of 60? I bet you could. Let's see what else you guys are working with. That's a fantastic start."

I stop by Charlie's desk, "What do you have? Nice, excellent. You're close . . . How close are you?"

Motioning toward Savannah and wanting to cultivate a community of learning I add, "She did the same thing you did, she found 25%." I nod at Savannah, "You're still working on that." Continuing with Charlie I ask, "Do you know how far 25% is from 35%?"

Charlie responds hesitantly, "10%."

"Yesterday we found 10%. Any remembrance about how to find 10%?" With that prompt I leave her to continue grappling with finding 10%.

As I confer briefly with multiple students, my intent is to affirm, encourage, and help prompt continued mathematical thinking and reasoning. I stop alongside Alizai and point at the blank above 50%, "That looks like a great start. If you're halfway through and this is 60, how many terabytes are there?"

When Alizai responds with 30, I encourage, "30. Okay, we'll throw that up there. So now you've got half. Can you find another one?"

Looking at Gabriela's work, I ask for clarification, "Is that 10%?"

When Gabriela confirms, I continue pointing at her paper, "There's 10 right there; 10% sounds really helpful if you already have 25%. Is that what you're doing? You got some really nice numbers on here."

I turn to Alizai sitting beside her, “Maybe you guys could share. Looks like you’re looking for similar things. Can you tell her why you were looking for those? Go for it guys.” I walk away knowing that partner talk may help each of them to clarify their own thinking.

I approach Nathaniel, “What are you thinking about? Hey, nice! I like where you put that. That’s excellent. Will you share that with everybody? I think that’s really helpful.” I am already thinking ahead of who I will ask to share, in what order, and for what purpose.

TIP

Use partner talk at strategic moments. Turning to talk to a partner gives students a chance to get their own thinking going, hear another’s ideas, and compare. It allows you a chance to circulate and hear what students are thinking.

To Miguel I ask, “What do you have going on? Nice. Once you’ve got that 25% . . . I like it. Wait, how’d you get that?”

When Miguel shares that he guessed, I respond, “Ah, good guess. Nice. So, then we could kind of check it. So, I really like your 25%. You guys both have 25%,” I say nodding to Miguel and Nathaniel. Again, I am working to build the sense of a learning community. I ask Nathaniel, “You haven’t decided what it is yet. Do you know what’s halfway between 0 and 30?”

When Nathaniel states it is 15, I affirm his thinking, “Bam, you got 15 terabytes. There you go. Nicely done.”

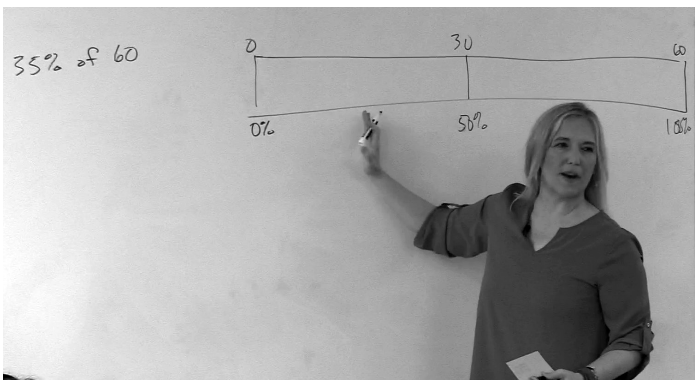
This first problem is not trivial, so taking time to confer with many students to help ensure a positive start for everyone is important.

I bring the class back together. “Alright, we have some really good stuff going on. Alizai, you started . . . I think when I looked at your paper, I think that you were right in the middle. What was right in the middle for the percent?”

When Alizai says, “50,” I repeat, adding the word percent,“50%.” I draw and label the halfway mark with 50%. “And then the two of you guys were talking, and you said, ‘If I’ve downloaded 50%, then how many terabytes is that?’ pointing to the 50% mark on the bottom of the bar and then the top of the bar where the corresponding terabytes will go.

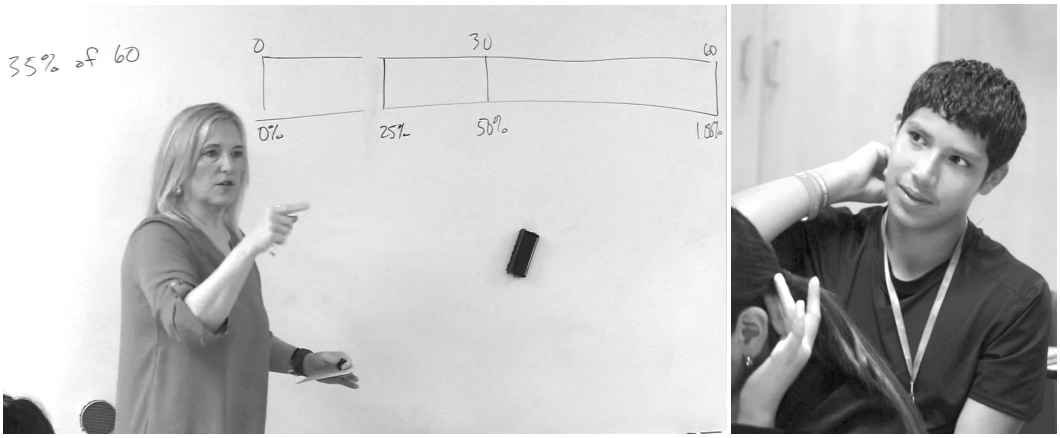
Gabriela responds, “30.”

I continue, “Everybody agree on that? I think a lot of you guys were sort of like, ‘I’m cutting that in half. I got the half.’ And then . . . Nathaniel, then you said . . . On your paper, you had some interesting other marks. Can you tell me the one that you had right in between here, right in the middle?”



Nathaniel answers confidently, “25.”

I draw and label 25% and ask, “Miguel, what did you guys say that was?” Notice I purposefully refer to them as a partnership thinking together.

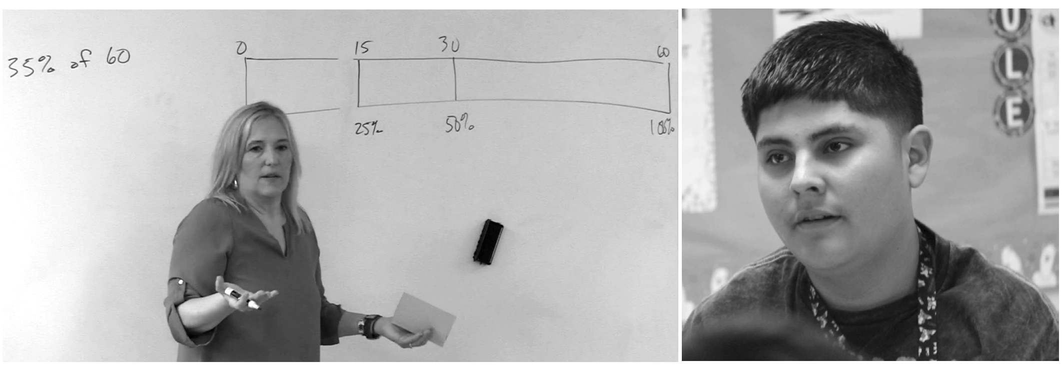


Miguel says, "15."

I say with a neutral expression, "You thought it was 15. How do you know that's 15 terabytes?" I write the corresponding 15 above 25%.

Nathaniel answers, "That's because it's half of 30."

I repeat, "It's half of 30. Does that make sense to everybody?"



I continue, "Hey, so far we've got some really helpful percents and terabytes that we can use. Then, Nathaniel, you had something on your paper. What was the next one that you marked? Do you remember?"

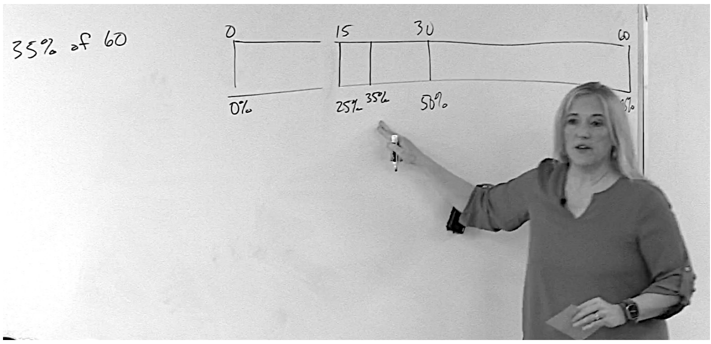
When Nathaniel answers, "35," I continue using my hands to gesture to locations on the percentage bar model. "Yeah, and I think your 35% was in between these two. Is that right? Did a lot of you guys put 35 somewhere between 25 and 50? Yeah, I thought you guys had. Charlie, you had something like that

TIP

A private signal, like a close-held thumbs-up, helps students think at their own pace rather than being rushed when they see other students raising their hands.

I believe. So, I think I saw something like 35% there." I mark the 35% location on the bar model. "And then, Charlie, you said something really, really important. When we were talking, you said these are how far apart?" I am attempting to draw in Charlie by publicly recognizing her reasoning.

Charlie says, "10%."



"10%," I repeat. "So, if we could just find 10%, Charlie's like, 'All we need is 10%.' You guys agree? Where's 10% up here? Gabriela, you had 10% up here."

"It's like near the 0," Gabriela says.

I gesture near the 0 and add a 10% line. "It's pretty near the zero. It's kind of over here. So, like if that's 10%, how are we going to find 10%? So, give me a thumbs up if you were finding 10%. I know you guys had a conversation about 10% back there. Alizai, did you get that far to talk about 10%? Not quite. Silas, what were you thinking about 10%?"

Silas says, "So, from what we did yesterday, it was like when you divided it by like 10, then it just like moved a decimal point. So, 60 divided by 10, I got 6." I make a mental note that at a later time I definitely want to do work with place-value shifts when scaling by 10, but for now we'll stay focused on the relationships between percentages and the whole.

With a neutral response, I add the 6 above 10% and ask, "Is 60 divided by 10, 6? Why is he dividing by 10?"

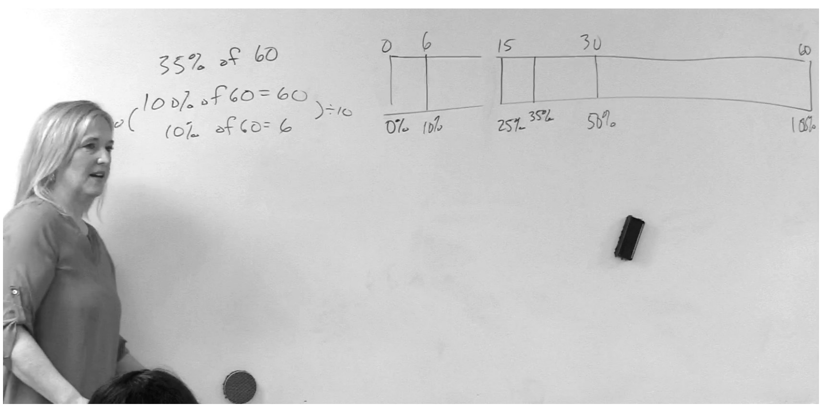
When Savannah answers, "Because it's 10%," I nudge, "Say more."

"10% like out of 60 or out of a 100," adds Savannah.

I ask, "What is the relationship between 10% and 100%?" I am wanting to focus the whole class to consider this fundamental relationship.

When Savannah replies, "There's one more 0," I try to get at the math that is happening. "Are you dividing by 10?"

"Yeah," says Savannah.



I then expound, "Yeah, you're talking about dividing by 10. We've got 100%. We're trying to find 10%. You guys are thinking, if I have 100% of 60, you guys said was 60. Then we could divide that 100% by 10, that's 10%. Bam. So, then 10% of 60, divide that by 10. You guys are like, 'That's 6.' Is 60 divided by 10, 6?" I make this thinking visible on the board by recording each of these equations and the scaling relationships.

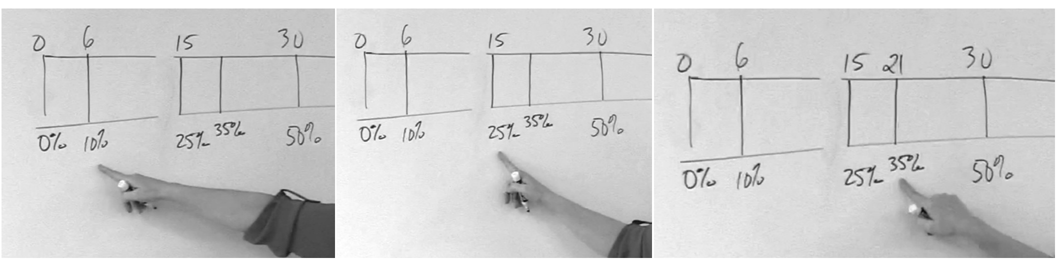
"Pretty cool," I say as I now erase those equations and return the focus to the bar model. "So, now we've got a bunch of pieces. Can we work with those pieces? De'Ambra, tell us how you're going to work with those pieces." De'Ambra looks at me silently, wide-eyed, and smiling. "No? You don't want to tell us how to work with those pieces?"

She shakes her head, "No."

"You sure?" I ask. I want to invite her to enter the conversation in a safe way, so I ask, "Can you tell us what pieces we do have?"

De'Ambra reads from the board, "10%, 25%, 35%."

I say, "We're looking for 35%. Can we use 10% and 25% to get 35%? How?" When De'Ambra shakes her head, "No," I respect her decision and try to keep her involved, by saying, "Do you want to choose someone? Who do you want to choose?"



TIP

When a student is hesitant to participate, look for low-risk ways to get them involved. You can ask a student to just get the thinking started or to just restate what they heard someone else say. You can offer to let them tell a friend their thinking and have the friend share their idea, or you can even ask them to select someone to help or answer for them.

De'Ambra chooses Savannah, who responds, "We did 6 plus 15 because they're at the top."

I ask for more detail for the benefit of all students trying to make sense of it, "Why are you adding 6 and 15?"

Savannah explains, "I don't know, because it's 10. I want to do 10 plus 25 is 35."

Motioning to the values and locations on our model, I repeat in context. "If 10% and 25%, add those together to get 35%, then you're like . . . What is 6 and 15 terabytes?"

A student says aloud, "21."

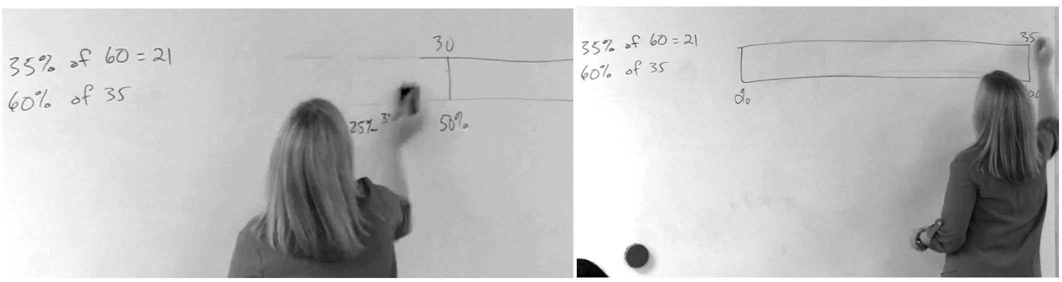
Responding to the class as a whole community of mathematicians I say, "You guys are thinking that might be 21. That kind of looks like 35% of 60 terabytes is

21 terabytes." I gesture with my arms again to help students focus on the whole percentage bar, drawing attention to the relative magnitude of 35%. "Nicely done, you guys. You guys used what you know to find one that was kind of tricky. Good job."

"Alright, next problem. What if I asked you for 60% of 35? 60% of 35. So, now everything changes because now we're not downloading . . . What are we downloading in total this time?"

Several students say, "35."

"Now, it's 35," I repeat. "So, the total is now 35 terabytes. Like, it's a whole new game. We're downloading a whole new set of stuff." I erase the bar model and draw a new one labeling the values in this problem. "We need 60%. What kind of pieces would be helpful to find 60% of 35?"



I circulate again, not as long this time. Several students, including Autumn, are finding 50%. Citlali and Enrique have a mark at 10%, and I briefly encourage them to keep working on finding how many terabytes are that 10%. Kaitlyn and Pressley explain they divided 35 to get 17.5. I ask, "Can you guys share that in a minute? Nice." I am intentionally planning for the sharing time.

I pull the class back together, “Autumn, will you start us off? Just that very first part?”

Autumn answers, “I would have to start with a 50%,” motioning with her hands making the dividing line for 50%.

I say, “You’re like 50%. Why is 50% helpful? Or keep going. I didn’t mean to interrupt . . . ” I think to myself that I should probably say less, as I draw the 50% mark on the model. Autumn continues, “Because it’s a halfway point.”

I then ask, “And so if you can find 50%, then you said we’d also need . . . ?” When Autumn says,“10%,” I add it to the model and ask, “Why is Autumn thinking that we need 50% and 10%? A lot of you guys did that. You’re like 50 and 10. Why 50% and 10%?”

Savannah states, “Because if you add it up together then you get 60%.”

TIP

A subtle and helpful move is to acknowledge students you conferred with, helping them know they are noticed and affirmed as mathers, that they are contributing even if they don’t share aloud this time.

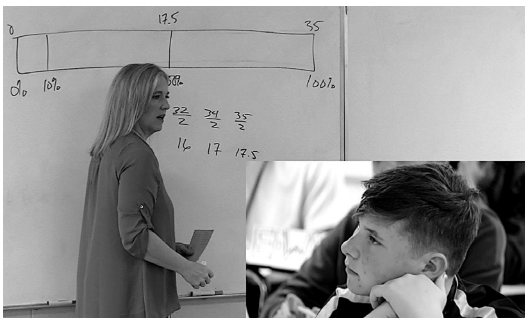
I reinforce this thinking, “And that’s what we’re looking for, right? We’re looking for 60%. Find 50 and 10. Bam.” Percentages are figure-out-able; find some chunks, and put them together! “You guys were looking for 10%. You and you and you,” and I point to various students, affirming their thinking. “Before we look for the 10%, let’s talk about the 50%. Kaitlyn and Pressley, you guys were talking about how you found 50%. Can you say more about that?”

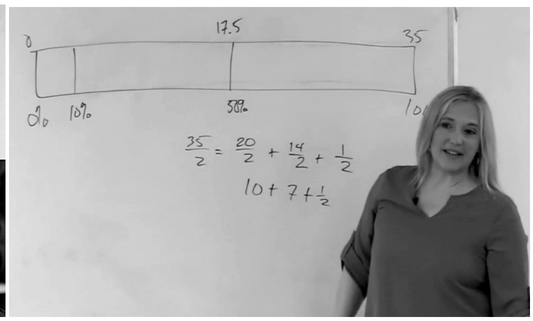
Kaitlyn shares that they divided 35 by 2.

I motion along the bar to illustrate these magnitudes and locations, “So, if you know the whole thing is 35, and you cut the 100% in half, you’ve got to cut the 35 in half.” The power in the percentage bar model is when students realize that what they do to percentages, they also do to the terabytes. “What is 35 divided by 2?”

“17.5,” answers Kaitlyn, and I record it on the board above 50% on the model.

We spend a few minutes sharing a couple of ways of finding half of 35, and I record the thinking of first Silas and then Edward as seen in the following photos. I purposefully pause here to tuck in some additional learning around decomposing numbers, fractions as division, and strategies to find half.





"Alright. Whew. So, we're downloading. We've got 50% is 17.5 terabytes. Nice. Now, we need that 10%. Savannah has an idea. Does anybody else have ideas about how we can find the 10%? Alright, so we definitely need to do some more work on 10%." I am filing this away in my mind to keep working on 10%.

Savannah explains, "I did 35 divided by 10, and I got 3.5."

I write this as an equation, 35 ÷ 10 = 3.5, and ask questions to engage the whole class. "You're like 35 divided by 10, and you got 3.5. What do you guys think? Why is Savannah talking about dividing by 10? What are we finding?" I pause. When I hear only one quiet response of "10%." I ask, "Why divide by 10 to find 10%?" Finding 10% is so foundational, I decide to land here for a bit.

Autumn joins the discussion. "So what Savannah said . . . we divide by 10 because it's that actual 10% of 35."

I use the bar model and gesture to give a sense of the values. "If here's 100%, and I got to find 10%, what is the relationship between 10 and 100? Divide by 10? One hundred divided by 10 is 10. Well, if we cut the percents into 10 chunks, we have to cut the terabytes into 10, also. And 35 divided by 10 is 3.5. Bam! There's 3.5 terabytes. Sweet!" I add the 3.5 above the 10% line.

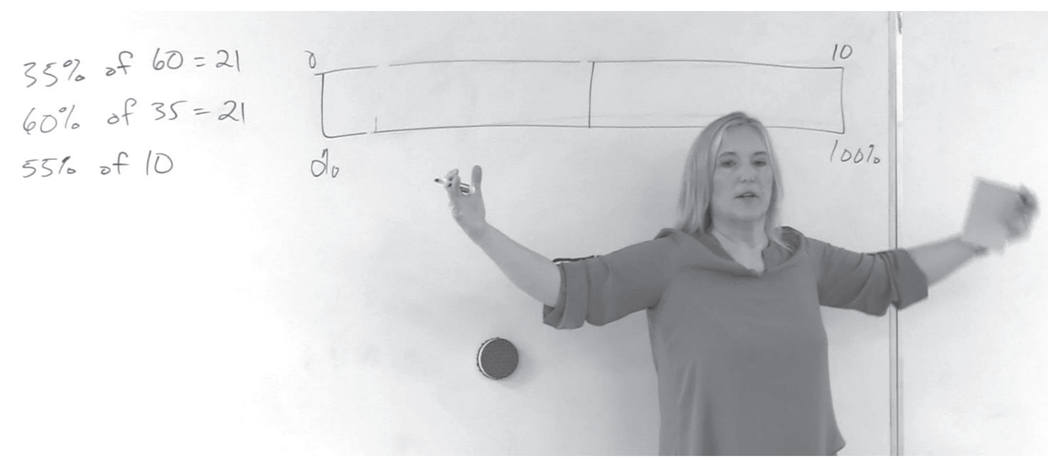
"So, then what is 60%? Give me a thumbs up if there's enough information now that you could tell us what 60% is. Joshua, what do you think? What is 60% of 35?"

When Joshua explains he added 17.5 and 3.5 to get 21, I point to the 17.5 and 3.5 on the model and their corresponding percent values and repeat, "Add those two together. Add the 50% and the 10%, and bam, you get 21. Cool!"

"Sit tall," I direct with a smile. "Nice job, guys. So, really we're going to work a little bit more on this 10% thing, but other than that, nice using what you know. Nice reasoning." I am intentional in celebrating what they have just done. It is not trivial. What you celebrate is what you value. They are using what they know to think and reason.

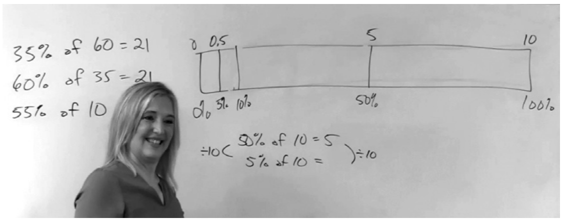
I continue, "Next problem. How about if we were going to do something like 55% of 10? Whew. 55 . . . "

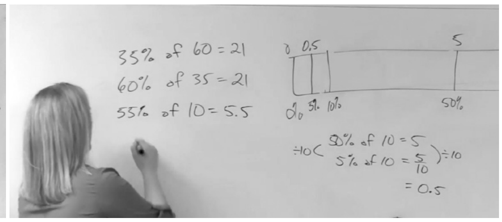
Savannah says, "Changes again."



With student input, we erase the old values and put the new values on the bar. Students work, I circulate and confer, and then I choose students to share.

I ask Abraham to share just the start of what he was thinking about, and Charlie pipes in after earlier saying she doesn't want to share but then volunteers. Along with input from other students, they find 50% and 5% and add them together, 5 + 0.5 = 5.5. We discuss the place value shift of finding the 5% from the 50%.





I take a breath. "Cool. All right, you guys, we're almost done. I bet you can do this one, ready? What is 10% of 55?" Again, I am giving students another opportunity to see and use the pattern of 10%. "Let me erase all this stuff. Hey, before we get started, what's the whole terabytes?"

Several students respond, "55."

I confirm, "Everybody's clear it's 55. And what are we trying to find?" With input from students we identify and label the 10% on the model. "Okay, 10%. We've been talking about . . . Hey, this is when we can practice our 10%-ness. Let's practice our 10%-ness. How are we finding 10%? Everybody's thinking. We're thinking." Students begin working, and I circulate.

TIP

A skilled facilitator listens to their students to hear what they are considering and nudges them forward in the moment. It's not enough to just "use what you know" willy-nilly, but rather to gain facility with important, major strategies that are useful long term.

When I reach Edward and Isaiah, I notice they are starting with 50%. I realize that this has been a helpful part to find in previous problems but is less helpful with this problem. I want to nudge them to think more about 10%, so I say, "I wonder if you could go straight to 10? I mean, all we need for this one is 10%. Like, you could find those others, but we just need 10%. How did we find 10% before?" The three of us then engage in a conversation reviewing the pattern that to find 10% of 100% you must divide by 10, so to find 10% of 55 you must divide by 10. When the boys show uncertainty about how to find 55 divided by 10, I suggest, "Let's have that conversation with everybody. You guys are awesome."

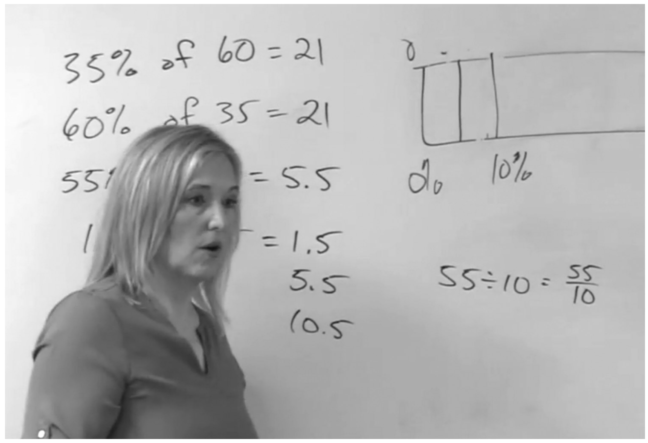
Back at the front, I begin, "Well, nice. Alright, not everybody's done, but let's chat." I reference my conversation with Edward and Isaiah wanting to reinforce their new understanding and restate how they concluded that since we divide 100% by 10 to know where the 10% goes, we must also divide the 55 terabytes by 10 to know the terabytes at 10%. At this point, I ask the class, "What is 55 divided by 10?" I write the equation on the board as I say, "And I can write it that way (55 ÷ 10), and I can write it that way $\left(\frac{55}{10}\right)$. What is 55 divided by 10?"

TIP

Responding with a neutral tone can help students realize they can stop looking to the teacher to be the only arbiter of truth. Students can gain confidence that they can reason in this logically consistent system called math and know they are correct.

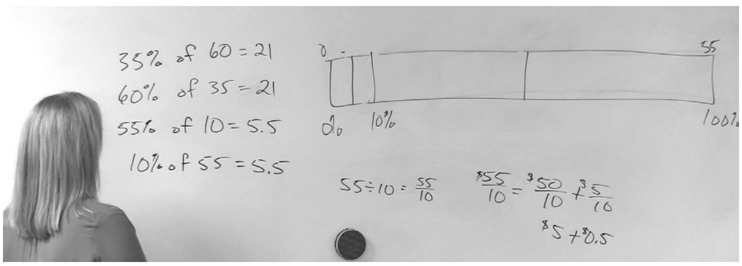
When a student replies, "1.5," I record it saying, "You're thinking it might be 1.5. Anybody think it might be something different? Silas?"

Silas says, "I think it's 5.5." I add it to the list and say in a neutral tone, "You think it might be 5.5? Let's get a third answer up here. Isaiah?"



When Isaiah answers, “10.5,” I say, “You think it might be 10.5?” and add it to the list of possible answers.

“How do we know what 55 divided by 10 is? Whew! Alright, let’s get some money.” I know that money is a great context to help students reason about division and decimal values. By asking the students to think about sharing $55 with 10 kids, the class is able to determine the amount would be $5.50, so 55 divided by 10 is 5.5. I then erase the two incorrect responses.



TIP

Context is helpful for more than just getting students engaged. Contexts like percentage download bars and money can help students realize the mathematical relationships as they use the context to reason. It’s not important that the contexts are real; they must help make the math realizable.

It is time to close out the lesson. I ask, “Okay, you guys up for one more problem? I’m a little curious. Did anybody notice any patterns just in these four problems up here?”

Savannah says, “Repeating the same number.”

I ask with an intrigued look, “Is there a number that’s repeating? That’s kind of weird. It’s probably just a fluke. It’s not intentional. Does anybody else see numbers that are repeating?’ Many students look surprised, not having noticed before. I restate what I hear students voicing aloud, “21, 21, 5.5, 5.5.” I point to these numbers on the board. “It’s probably just an accident,” I shrug.

Savannah observes, “You flipped them.”

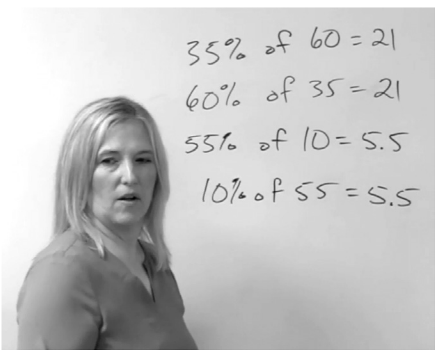
I ask, “I flipped what? Before you say anymore . . . Savannah said something about flipping. Are there some numbers flipped?" There is a mumbling of, "Oh, yeah!" among the class.

I continue, holding back a smile and staying neutral, "21, 21, 5.5, 5.5. There's no flipping going on. They're the same. Krysta, do you see something?"

Krysta replies, "It's like the same problem, it's just formatted differently."

To which I respond with one of my favorite teacher moves, "Say more." This sends the message that it's okay to just get started and then to say more, giving students the sense that we're building this together. They don't have to have the whole explanation at once.

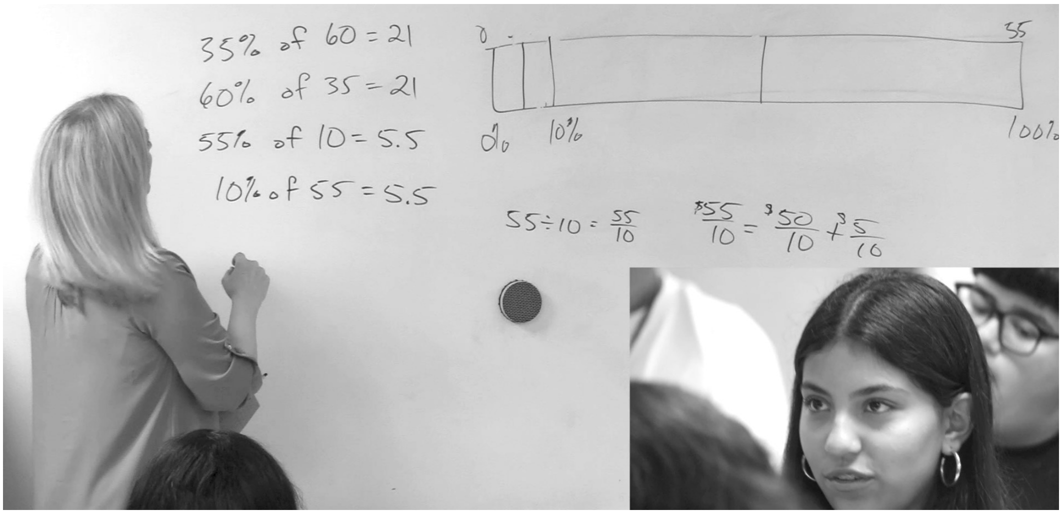
Krysta adds on, "So, like the 55% and the 10, it makes that 5.5. But if you write it the other way, it'll still make the same equation because they're still the same numbers. It's just you have to work differently."



TIP

Knowing your content and students well helps you adjust on the fly. I tweaked the last problem from 37 to 26 because students had already struggled halving an odd number, and I wanted the focus to be on the commutative property. We'll return to halving later.

I ask in a puzzled tone, "You think you'll get the same answer? Like, if you do 35% of 60, you're saying you'll get the same answer as if you do 60% of 35? No way!" I express genuine interest and add, "I just chose numbers that would work for today. There's no way that it would actually work all the time that way. That would be silly." I pause for effect. "Let's do one more problem. How about if I asked you, 37% of 50?" I quickly adjust, based on how the string has gone so far. "You know what, I'm going to change that. What if I were to ask you 26% of 50? Pause for a second. I wonder . . . I wonder . . . Okay, everybody thinking, 26% of 50. Go."



Kids start thinking, and I begin to observe as they work. I know we are running short on time, so I stop them. "I'm going to leave you thinking about that one. Did anybody think about 50% of 26? I am going to wonder if you could."

Students walk out of the door that day with many curious looks on their faces. I'm excited to see what they'll bring tomorrow when we keep gaining clarity the next day.

Scan the QR code on this page to watch me facilitate this Problem String and to download a facilitation guide.

TIP

This is one of a series of Problem Strings to develop percentages. The Problem String we did the day before is in Chapter 5.

Classroom video and facilitation guide.

https://qrs.ly/ewha2hn

To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

FREQUENTLY ASKED QUESTIONS

Q: Are the students done learning about percentages after this lesson?

A: No, students will need more experiences to truly own the relationships they were using in this Problem String. But students are not *only* getting answers (and they are), but they are *also* learning mathematical behaviors like diving in and using what they know to logic their way through the problem, putting words to the relationships they are using, and representing their thinking on a percentage bar. They are also learning to use the commutative property with percentages and to reason proportionally.

(Continued)

(Continued)

Q: Why do you call this a Problem String? Isn't it just a really long Number Talk?

A: A Problem String is a powerful instructional routine. It is a series of intentionally sequenced problems where the teacher elicits and represents student strategies and crafts discussions between each problem. You will read more about this remarkable teaching tool throughout the book.

THE DEVELOPMENT OF MATHEMATICAL REASONING

Our goal in math instruction should be to help students develop mathematical reasoning about and with the content. This is not some fuzzy "think better" game. Instead, it is helping students learn to logically reason using mathematical concepts, strategies, models, and properties. Reasoning mathematically is solving problems using what you know and in the process building more and more real math. It is not just using what you know to solve problems but using what you know to figure out math you are still in the process of learning.

Reasoning mathematically is solving problems using what you know and in the process building more and more real math.

The Development of Mathematical Reasoning graphic defines a hierarchy of reasoning domains consisting of increasing levels of sophistication and simultaneity (Harris, 2025). It represents the high-level hierarchy of milestones students must progress through to develop reasoning-based proficiency.

To reason through a problem, a student must grapple with multiple levels of complexity simultaneously. As students' brains develop, they create schema—ways of structuring their mental maps so that they can focus on the big picture or drill down to the specifics. Because they own this relationship map, their ability to grapple with multiple things will increase, leading to more efficiency and understanding.

As students develop their brains, they create schema—ways of structuring their mental map so that they can focus on the big picture or drill down to the specifics.

Sophistication includes the level of simultaneity employed as well as the complexity of the mathematical ideas at play.

Consider the following example of increasing simultaneity and sophistication when a student begins to reason multiplicatively. A student reasoning additively about the multiplication problem 5×9 might add 9 over and over, until they have added five 9s. A student reasoning multiplicatively might realize 10×9 is 90, and 5×9 is half of that, so 5×9 is also 45. Ideally, a student's brain will travel that path so often that it becomes a well-traveled path and those relationships become compressed. The equation $5 \times 9 = 45$ becomes automatic, we might say "memorized," but those connections can also be brought to mind and used when working with other problems like 5% of 9 (find 10% and cut it in half) and 50% of 9 (cut 100% in half) and also 15% of 9, 15% of 90, and 5% of numbers like 64 or 89.

TRY IT

Use the strategy Pam's students were developing to find 5% of 64 and 15% of 89 by thinking about 10% of 64 and 10% and 5% of 89.

The goal of Grades 6–8 is creating and expanding the mental paths of noticing patterns, using those patterns, and making generalizations to use Additive Reasoning, Multiplicative Reasoning, and Proportional Reasoning with rational numbers. Students should also begin to develop Functional Reasoning.

THE COUNTING STRATEGIES DOMAIN

When young students begin to solve problems, they use Counting Strategies.

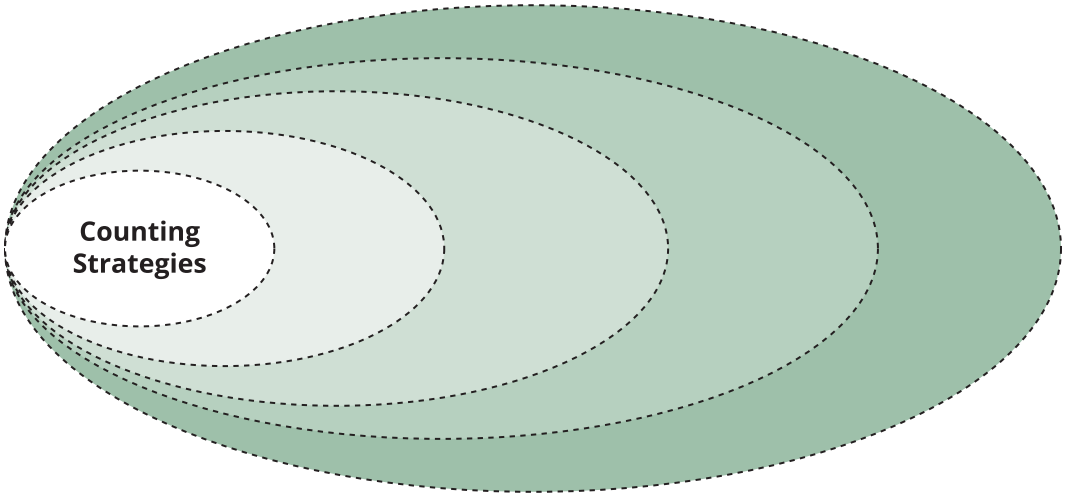
Using Counting Strategies entails solving problems with one-by-one counting. Using Counting Strategies is more than just being able to say the counting sequence. It's about solving problems and, while solving, considering the numbers involved as sets of ones. We can solve addition, subtraction, and even multiplication and division problems counting one by one.

The Counting Strategies domain is the foundation of everything else. In Grades 6–8, we need to help students develop more sophisticated ways of dealing with additive, multiplicative, and proportional situations. We need to help students learn to grapple and deal with bigger jumps than one at a time.

TIP

Do not despair if some of your students are still solving additive, multiplicative, or proportional problems by counting. This book will help you help these students.

FIGURE 1.1 ● The First Domain of Sophistication in Mathematical Reasoning



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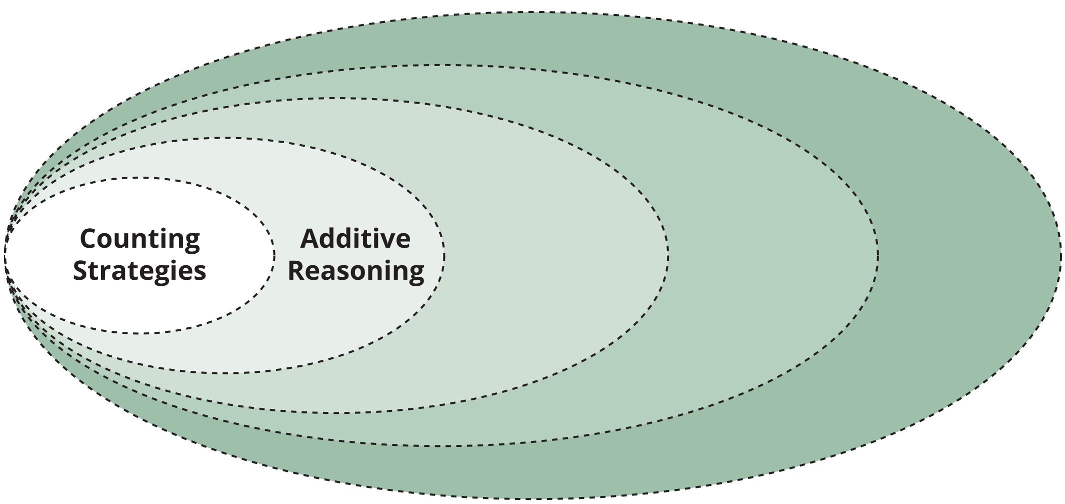
THE ADDITIVE REASONING DOMAIN

Additive Reasoning is characterized by thinking in bigger jumps of numbers than one at a time.

FIGURE 1.2 ● The Second Level of Sophistication in Mathematical Reasoning



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An additive reasoner considers numbers simultaneously as sets of ones and combinations of other sets of numbers. It's about composing and decomposing numbers in additive chunks, solving addition and subtraction problems by considering the place values and magnitudes (sizes) of the numbers. Building on the additive reasoning from Grades 3–5 into larger, smaller, and more complicated numbers like decimals, fractions, ratios, and integers is a major goal of Grades 6–8. This means that the focus is always working toward students making sense of the relationships between numbers. This takes time, effort, and many experiences.

TABLE 1.1 ● Comparing Counting and Additive Reasoning

Counting Strategies Within an Algorithm	6. 7, 8, 9 8. 9, 10, 11, 12 9. 10, 11, 12, 13, 14 16.89 +2.35 19.24
Single Digit Additive Strategies Within an Algorithm	7 + 2 = 9 9 + 3 = 10 + 2 = 12 9 + 5 = 10 + 4 = 14 16.89 +2.35 19.24
Additive Reasoning	0.11 + 2.24 = 2.35 16.89 17 19.24 or 16.89 + 2.35 +0.11 −0.11 17 + 2.24 = 19.24

TRY IT

Solve the problem 62 – 15 using all three kinds of reasoning shown in Table 1.1.

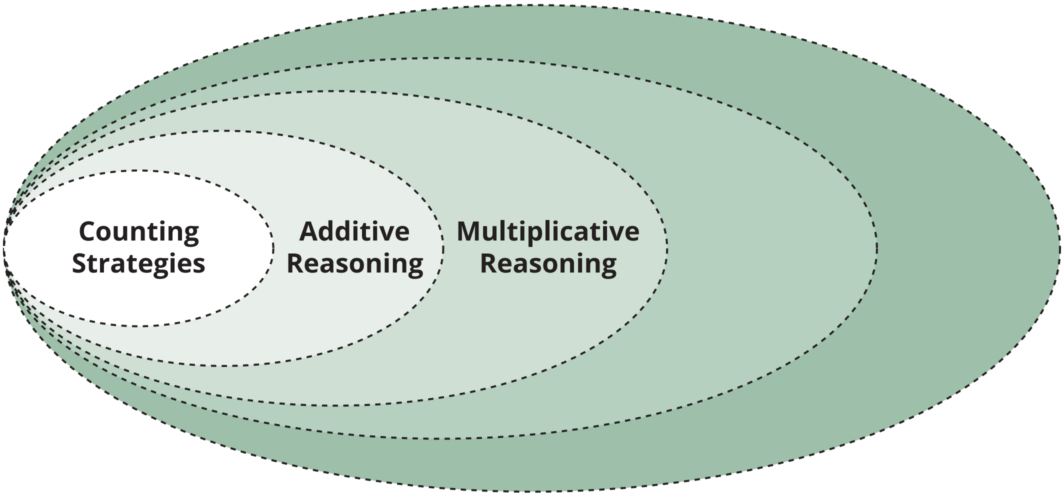
A goal of Additive Reasoning is to work toward using fewer, bigger jumps when solving problems.

When learning to solve addition and subtraction problems using Additive Reasoning, it's far more than getting answers—it's building a sense of when to add and when to subtract and the relationship between addition and subtraction.

THE MULTIPLICATIVE REASONING DOMAIN

Multiplicative Reasoning is more mentally sophisticated than Additive Reasoning.

FIGURE 1.3 ● The Third Level of Sophistication in Mathematical Reasoning



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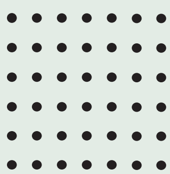
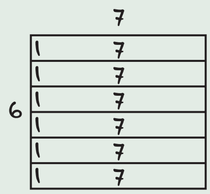
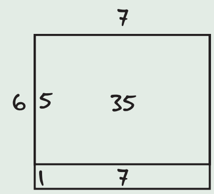
Solving a multiplication or division problem using Additive Reasoning looks like adding or subtracting one group at a time. Solving these problems using the more sophisticated Multiplicative Reasoning means using more than one group at a time, grouping the groups. A person is reasoning multiplicatively when considering the number of groups, the number in each group, and the total all at the same time. The shift from reasoning additively to reasoning multiplicatively is an immense development for the learner.

As shown with the ovals in Figure 1.3, Multiplicative Reasoning is built on and based upon Additive Reasoning. Because of this, it is essential that Grades 3–5 instruction continues to strengthen Additive Reasoning for additive situations—moving students beyond Counting Strategies—and develops Multiplicative Reasoning for multiplicative situations—moving them beyond skip-counting, even when they're getting correct answers.

We can compare the first three domains by using arrays to illustrate the different mental actions a student could take to solve the problem 6 × 7. In Table 1.2, using a Counting Strategy looks like counting 1 by 1 to 7 and keeping track that you do that six times. Using Additive Reasoning looks like adding 6 groups of

7: 7, 14, 21, 28, 35, 42. Using Multiplicative Reasoning can look like chunking the area into two chunks: $5 \times 7 = 35$ and one more group of 7, $35 + 7 = 42$.

TABLE 1.2 ● Comparing Counting, Additive, and Multiplicative Reasoning

COUNTING STRATEGIES (ONE BY ONE)	ADDITIVE REASONING (ONE GROUP AT A TIME)	MULTIPLICATIVE REASONING (GROUP THE GROUPS)
	6×7	
		

TRY IT

For a similar problem, 7×8, represent the three levels of reasoning as shown in Table 1.2, Counting Strategies, Additive Reasoning, and Multiplicative Reasoning.

When learning to solve multiplication and division problems using Multiplicative Reasoning, it's far more than getting answers—it's building a sense of when to multiply and when to divide and the relationship between multiplication and division. It's also important to use rectangular models to develop the sense of two-dimensional measurement we call area and three-dimensional measurement we call volume.

In early development of Multiplicative Reasoning, students reason additively—considering one group at a time. This looks like equal jumps on a number line and one row (or column) at a time in an array.

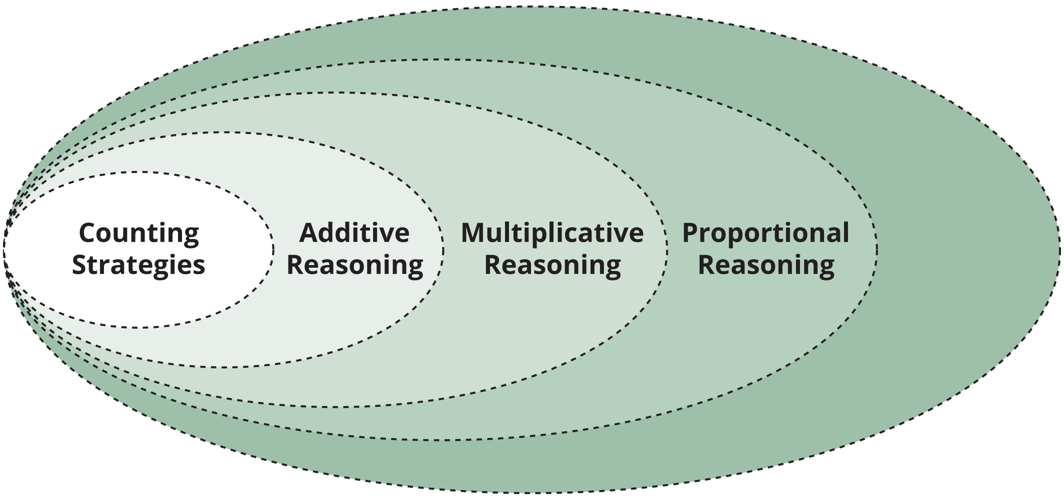
Once they begin to understand what multiplication means, students start grouping the groups into partial products. Single-digit multiplication solvers work with smaller groups, then bigger groups in multi-digit multiplication. In Multiplicative Reasoning, a goal is to use fewer, bigger chunks.

Another part of multiplicative reasoning is realizing that double the number of groups means double the product and that half the groups means half the product. Students also need to grapple with the reciprocal relationship inherent in multiplication—that when you halve the number of groups, you must have double the number in each group to keep the same product. For example, to find 12×4, 12 groups of 4, you can find 6 groups of 8. And you could also find twice as many groups that have half as many in each group, like 24 groups of 2.

THE PROPORTIONAL REASONING DOMAIN

Proportional Reasoning is more mentally sophisticated than Multiplicative Reasoning.

FIGURE 1.4 ● The Fourth Level of Sophistication in Mathematical Reasoning



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As many as 90% of adults in the United States are not reasoning proportionally (Lamon, 2020). The vast majority of those same adults took classes in high school algebra, geometry, and more. This means they were getting answers, but because they did not have the necessary building blocks of reasoning, they were not able to reason more sophisticatedly.

As many as 90% of adults in the United States are not reasoning proportionally.

In Proportional Reasoning, students learn to reason about ratios and proportions—parsing out the differences between fractions, ratios, and rational numbers.

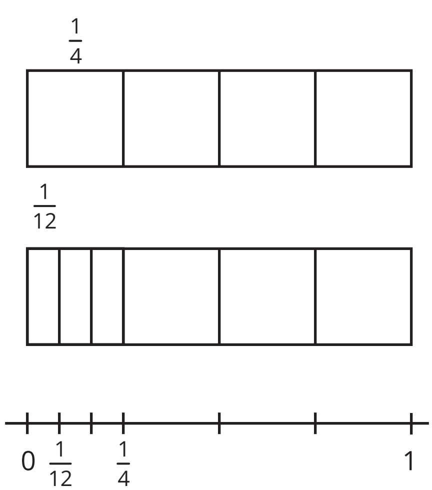
There are five interpretations of rational numbers, and they are all interrelated (Lamon, 2020). This means, for example, we can consider $\frac{3}{4}$ as any of the following:

- Part-whole: 3 out of 4
- Measurement: $\frac{1}{4}$ as a location on a number line so $\frac{3}{4}$ as jumps of $\frac{1}{4}$, three $\frac{1}{4}$s, $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=3\left(\frac{1}{4}\right)$
- Operator: $\frac{3}{4}$ of something
- Quotient: 3 divided by 4, $3 \div 4$
- Ratio: 3 to 4, 3 : 4

In Grades 6–8, as we teach rational numbers, it is important to help students develop a well-rounded sense of what rational numbers are, with these five interpretations. This means students are reasoning with the meanings of fractions, ratios, quotients, and so on and using the major strategies to solve problems.

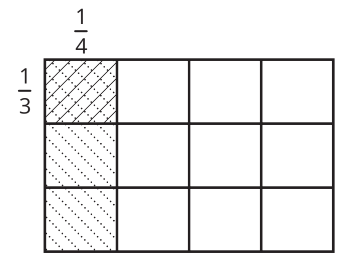
Chapter 4 discusses each of these five interpretations in detail, diving into how to start building each one in students. We also want to build a few important precursor ideas:

- Dealing with same-size pieces (denominators) and how the sizes of pieces relate to each other
- Understanding the product of unit fractions, that $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$, in two ways
    - This means that students reason about $\frac{1}{3}$ of $\frac{1}{4}$ by thinking about cutting $\frac{1}{4}$ into 3 equal chunks, which means that each $\frac{1}{4}$ of the whole is divided into 3 equal chunks so the whole is cut into 12 equal chunks, $\frac{1}{3} \times \frac{1}{4}$ is $\frac{1}{12}$. This is analogous to cutting a length of 1 into 4, then that $\frac{1}{4}$ length into 3, for a length of $\frac{1}{12}$.



- Students also need a sense of the area of $\frac{1}{3}$ by $\frac{1}{4}$ of a rectangle. If you have $\frac{1}{4}$ of a rectangle and you find $\frac{1}{3}$ of that fourth, you have 1 out of the 12 pieces.

$$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$



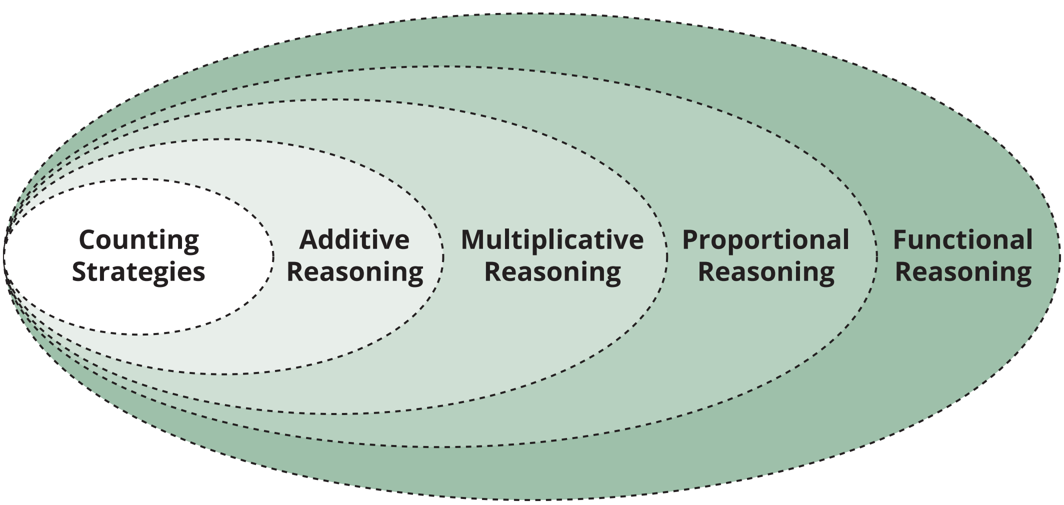
TRY IT

Using the example of $\frac{1}{3}$ of $\frac{1}{4}$, represent the two ways to reason about $\frac{1}{2}$ of $\frac{1}{3}$.

THE FUNCTIONAL REASONING DOMAIN

Functional Reasoning builds on the previous domains and is more mentally sophisticated.

FIGURE 1.5 • The Full Spectrum of Mathematical Reasoning



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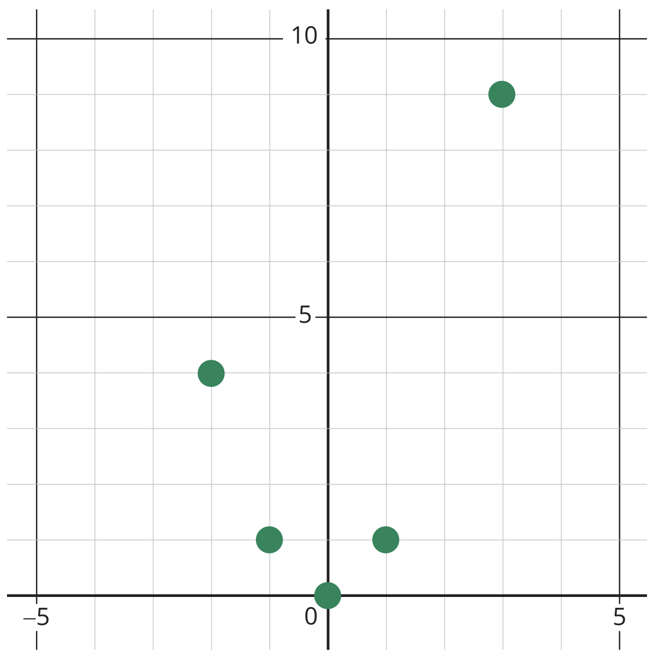
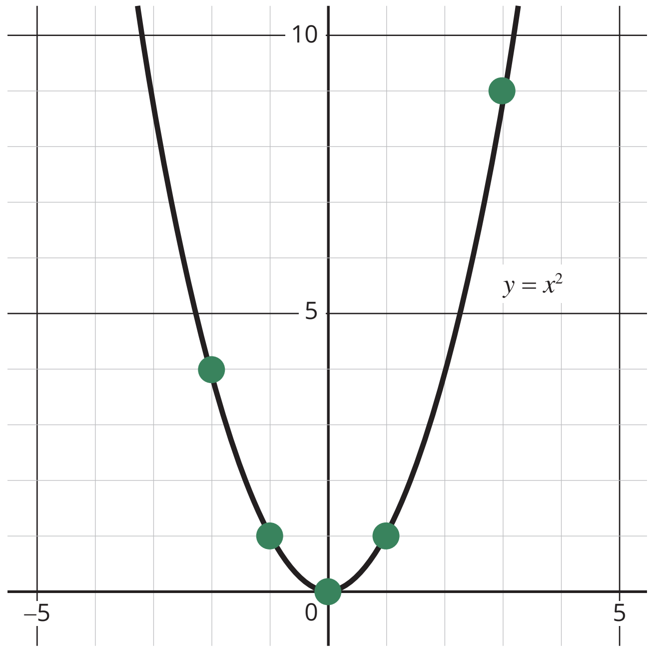
Whereas proportional reasoning is all about two *numbers* varying in tandem, functional reasoning is all about the relationship between *whole sets of numbers*. We could call functional reasoning bivariate covariation, because it is not just reasoning about functions but also relations, tables, graphs, and transformations.

Functional Reasoning involves more than just input-output machines. A function machine treats values as discrete: put in a value, get out a value. To build Functional Reasoning, we help students think about the behavior of relationships across a range of values, not just as a discrete set of values like in an input-output table. Functional Reasoning means thinking about patterns over an interval, patterns that often continue indefinitely.

For example, if x is any real number and y is the square of all those xs, we can consider some of the numbers that fit this requirement as a table of paired values or as isolated points on a coordinate plane. We can also represent this relationship as a continuous graph and as an equation $y = x^2$—both of these representations connect all possible xs to all possible ys.





A strong functional reasoner might see any one of these representations and think about how varying the *x*-value will affect its connected *y*-value. Functional Reasoning also includes considering how certain changes to the connection between *x* and *y* affect any one of those representations: What if, after squaring *x*, we add 8? How does that change the table of values, graph, and/or equation? What if we square the *x* then scale by 1.5? A strong functional reasoner considers the effect of these changes not just for a few discrete values but for all *x*s between those values as well as over the entire domain.

To learn more about the domains of reasoning, see my book *Developing Mathematical Reasoning: Avoiding the Trap of Algorithms* (Harris, 2025).

Developing Mathematical Reasoning: Avoiding the Trap of Algorithms.

https://qrs.ly/hkha2hr

SPATIAL, ALGEBRAIC, AND STATISTICAL REASONING

In addition to the five hierarchical domains in Figure 1.5, there are three longitudinal domains: spatial, algebraic, and statistical. I refer to them as longitudinal, because unlike the five hierarchical domains, they don't build off other domains of reasoning. Each of these three domains should be developed in concert with the hierarchical five.

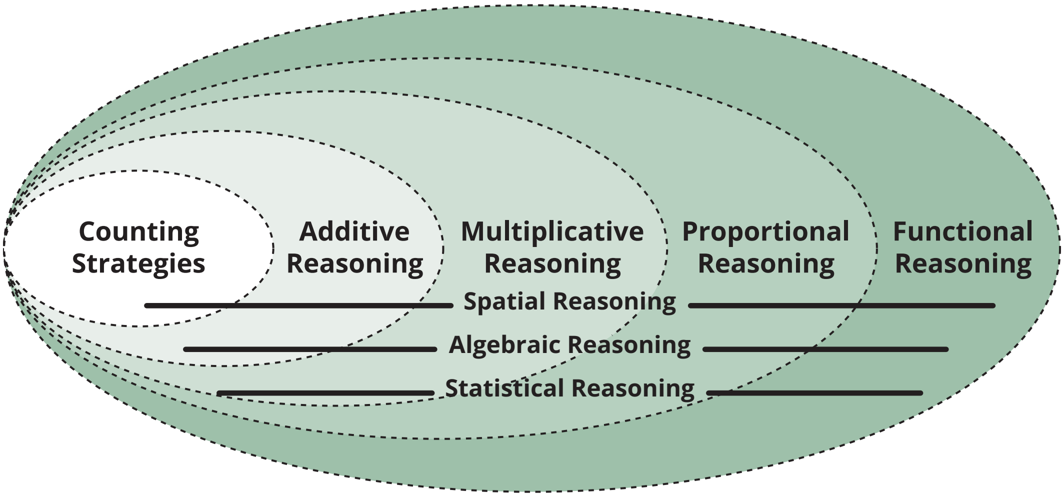
For example, Spatial Reasoning should be in development as students are moving all the way from Counting Strategies (1 : 1 tagging) to Multiplicative Reasoning (area models), to Functional Reasoning (distance and area defined by functions). Spatial reasoning is all about visual, geometric relationships of shapes, dimensions, measurement, location, graphs, and trends.

Algebraic Reasoning is all about generalizing and working with these generalizations. In counting, this might be that whether you have 5 dogs, 5 marbles, 5 balloons, or 5 anything else, the quantity is 5. In addition, this could be that you can add 10 to any number and the digits in the ones place don't change but another 10 is added resulting in a change in the tens and then hundreds place. In multiplication, that might be that 5 multiplied by anything is half of that thing multiplied by 10. In solving a proportion, that might be figuring out what value maintains the same multiplicative relationship with 8 that 3 has with 6. In later mathematics, we generalize relationships using variables and call it algebra.

Statistical Reasoning concerns data, inferences, and predictions. This is the ability to draw useful information from data sets and create meaningful conclusions. It is also the ability to recognize when someone has manipulated factual data to favor their biased conclusions. At a basic level, Statistical Reasoning includes representing data in a graph (the number of people in students' families), discussing the data (many of us have four people, some have fewer, and some have more), and possible changes (how would the graph change if we had a student move in with five people in their family?). We also look at the measures of center, the distribution, and the range and how changes in the data affect each and vice versa. Reasoning about data is becoming increasingly important in all of our lives and the beauty is that we can use data to help build the rest of mathematical reasoning (Boaler & Williams, 2026).

Any lesson or routine intended to build reasoning must be conscious of how the material intersects the five hierarchical domains of reasoning with the longitudinal three. For example, spatial array models for students learning Multiplicative Strategies will not be very effective if those students do not yet have the spatial reasoning to appreciate the measurement meanings behind area models.

FIGURE 1.6 ● The Full Spectrum of Mathematical Reasoning With Longitudinal Domains



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FREQUENTLY ASKED QUESTIONS

Q: This Development of Mathematical Reasoning (DMR) graphic looks similar to standards progressions. Do your strategies line up with my required state standards?

A: In short, yes. At first glance, the DMR graphic looks like the normal progression of school math: count first, then add, then multiply, and so on. Grade-level standards often suggest when students should have proficiency with specific operations. Notice how the goal shifts when the emphasis is on reasoning rather than answer-getting. It is not enough to say that since a student can solve a proportion, they have mastered proportional reasoning. Since students can solve proportions without ever reasoning proportionally, we want to put the emphasis on advancing reasoning rather than just answer-getting!

Q: Does this book have everything I need to satisfy my state's standards?

A: The emphasis in this book is on building a pattern of reasoning in mathematics. We look at major ways students can develop the domains of reasoning. However, there are other aspects of grade-level instruction that need to be included, such as data, probability, geometry, and so on. As you incorporate other requirements into your lesson plans, remember that math is figure-out-able, and you can teach it that way!

MAJOR STRATEGIES

These are specific approaches to problem solving that take advantage of the human mind's natural pattern-finding abilities to build a student's mathematical understanding and empower them to solve problems quickly and efficiently. We can cultivate and train mathematical intuition that allows students to engage in *math*-ing.

Unlike when memorizing algorithms, learning strategies are synergistic. Where each new algorithm is another series of steps to potentially misremember and confuse (is it invert/multiply or cross multiply and divide?), strategies are mutually reinforcing. The more relationships you own, the more strategies you learn, which in turn builds more relationships.

Unlike when memorizing algorithms, learning strategies is synergistic. Where each new algorithm is another series of steps to potentially misremember and confuse (is it invert/multiply or cross multiply and divide?), strategies are mutually reinforcing.

As SanGiovanni et al. (2022) suggest in their *Figuring Out Fluency*, "Real fluency is the ability to select efficient strategies: to adapt, modify, or change out strategies to find solutions with accuracy" (p. 4).

This book will help you learn the major strategies for addition, subtraction, multiplication, division, and proportional reasoning and how to teach them.

Note that while, for example, there are four major strategies for division and there is generally only one division algorithm traditionally taught, it does not take four times as long to teach four strategies as it does to teach one algorithm. Each subsequent step of an algorithm breeds confusion or presents another opportunity for error, whereas every major strategy builds off the one before it, bringing increased clarity and proficiency. Every subsequent major strategy learned makes mistakes with a previous one less likely.

Every subsequent major strategy learned makes mistakes with a previous one less likely.

Finally, because strategies create relationships (through context-driven understanding), student retention of learning is far higher than with algorithms, which tend to create disconnected, easily lost islands of procedure (Jensen & McConchie, 2020).

FREQUENTLY ASKED QUESTIONS

Q: Are you advocating for direct instruction or inquiry?

A: I'm advocating for a shift in goals, from mimicking algorithms to developing mathematical reasoning. With that new goal in mind, how to teach becomes clearer: Use guided inquiry for everything that is logical knowledge and clearly tell the bit that is social/conventional knowledge. Teachers have clear mathematical goals: Help students grapple long enough, guide students to important generalizations through purposefully crafted discussions, anchor learning, and keep building on that learning to move the mathematics forward using open-enough tasks that all students continue to have access to and continue to be challenged by. By doing this, students are solving problems not just correctly and efficiently but also more sophisticatedly. Students will be more successful longer. The rest of the book details how to do this.

Conclusion

The purpose of math class is to develop mathematical reasoning, not mathematical answer-getting. What we need are not mere calculators but thinkers and do-ers of mathematics. Our role as teachers is not to have students rotely mimic algorithms that only provide answers to problems but to guide and support students as they develop as mathers (Peart Crayton, 2026). We can help students realize that they can use what they know to solve problems. Math is figure-out-able!

Discussion Questions

1. What was your experience as a student in math class? Did you more often reason through problems, using what you know? Did you rote-memorize and mimic your teacher? How do you think that impacts the way you teach?
2. What do you see as the purpose of math class? How do you want that to impact your teaching?

3. What are the domains of mathematical reasoning? What is your understanding of them for the grade you teach?
4. Were you taught to reason additively? Multiplicatively? Proportionally?
5. How do you think your students are solving an addition or subtraction problem, by mimicking an algorithm or reasoning additively? What does that look like?
6. How do you think your students are solving a multiplication or division problem, by mimicking an algorithm or reasoning multiplicatively? What does that look like?
7. How do you think your students are solving proportions, by mimicking an algorithm or reasoning proportionally? What does that look like?
8. What's the difference between an algorithm, a strategy, and a model? The book will further differentiate between these, but for now, how are you thinking about the differences?