1

Numbers and Operations

Grades PreK–2

The concepts and skills related to numbers and operations are a major emphasis of mathematics instruction in prekindergarten through Grade 2. Over this span, the small child who holds up two fingers in response to the question "How many is two?" grows to become the second grader who solves sophisticated problems using multidigit computation strategies. In these years, children's understanding of numbers develops significantly. Children come to school with rich and varied informal knowledge of numbers. During the early years teachers must help students strengthen their sense of numbers, moving from the initial development of basic counting techniques to more sophisticated understandings of the size of numbers, number relationships, patterns, operations, and place value (NCTM, 2000).

STRATEGY 1: Encourage young children's exploration and understanding of relationships among numbers.

NCTM Standard

Understand numbers, ways of representing numbers, relationships among numbers, and number systems.

What Research and the NCTM Standards Say (NCTM, 2000)

Counting is a foundation for students' early work with numbers. Young children are motivated to count everything from the treats they eat to the stairs they climb, and through their repeated experience with the counting process, they learn many fundamental number concepts (Ginsburg, Klein, & Starkey, 1998). They can associate number words with small collections of objects and gradually learn to count and keep track of objects in larger groups. They can establish one-to-one correspondence by moving, touching, or pointing to objects as they say the number words (Baroody & Wilkins, 1999). They should learn that counting objects in a different order does not alter the result, and they may notice that the next whole number in the counting sequence is one more than the number just named. They often solve addition and subtraction problems by counting concrete objects, and many children invent problem-solving strategies based on counting strategies (Fuson, 1988).

Classroom Applications

In this vignette (adapted from Baroody & Wilkins, 1999), we listen as five-year-old Tammy is being encouraged by her father to build on her understandings of counting to compare numbers. The vignette is set in Tammy's home as much of children's early number sense comes from one-to-one interactions with caretakers and siblings. However, similar learning experiences can be provided in classrooms by kindergarten and prekindergarten teachers.

Tammy is playing the card game War with her father. War is normally a two-player game that is played by dealing out all of the cards (in this case, face cards and jokers were removed), facedown, to each of the players. Both players then, simultaneously, turn over the top card of their pile, and the player with the highest number wins both cards. If there is a tie for the highest, then both players put a second card facedown and simultaneously turn up a third card. The player with the highest number wins all the cards played in that round. If there is again a tie, this tie-breaking procedure is repeated until one player wins or one or both players run out of cards.

During the game, Tammy drew an 8 and her father drew a 6. Unsure which number was larger, Tammy said, "Wait a minute," and then got up and went to the channel selector of the TV. She looked up each number on the channel selector and concluded, "Eight is higher than six." Soon after, a 7 and an 8 came up. She again went to the channel changer to determine the larger number. Later, a 9 and an 8 came up. "Which is bigger, Daddy?" she asked. "What do you think?" her father said. Tammy returned to the channel selector and concluded, "Nine is much bigger." Several plays later, a 9 and an 8 came up again. This time Tammy counted the spades on her 9 card ("one, two, three, four, five, six, seven, eight, nine") and took the cards because nine followed eight when she counted.

Tammy has been engaged by her father in an activity that is—in her eyes—both meaningful and challenging. If teachers and parents deliberately create such opportunities and provide such tools—a clock or some sort of number line are other examples—children can learn to become quite resourceful in their mathematics solutions.

Precautions and Possible Pitfalls

Young children are often able to count accurately to reasonably large numbers. However, such counting can be more of a chanting of number words than a purposeful one-to-one correspondence with objects, and with actual objects, such children frequently overcount, exhibiting little awareness of cardinality. It is important that teachers and parents help children—by providing opportunities to count distinct objects, in a purposeful one-to-one fashion and in context—move beyond such chanting to a deeper understanding of numbers.

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STRATEGY 2: Encourage young children's understanding of addition and subtraction and how they relate to each other.

NCTM Standard

Understand meanings of operations and how they relate to one another.

What Research and the NCTM Standards Say (NCTM, 2000)

As students in the lower grades work with complex tasks in a variety of contexts, they also build an understanding of operations on numbers. Appropriate contexts can arise through student-initiated activities, teacher-created stories, and in many other ways. As students explain their written work, solutions, and mental processes, teachers gain insight into their students' thinking. An understanding of addition and subtraction can, for example, be generated when young students solve "joining" and take-away problems by directly modeling the situation or by using counting strategies, such as counting on or counting back (Carpenter & Moser, 1984). Students develop further understandings of addition when they solve missing-addend problems that arise from stories or real situations. Further understandings of subtraction are conveyed by situations in which two collections need to be made equal or one collection needs to be made a desired size. Some problems, such as "Carlos had three cookies. María gave him some more, and now he has eight. How many did she give him?" can help students see the relationship between addition and subtraction.

Classroom Applications

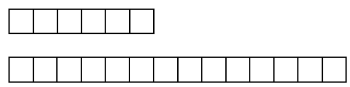
In this vignette (adapted from Barnett-Clarke, Ramirez, Coggins, & Alldredge, 2003, Case 1), we listen as Ms. Santi's second graders are being encouraged to deepen their understandings of subtraction.

Ms. Santi's students had been, in prior weeks, practicing addition and subtraction fact families and working on problem-solving skills involving both addition and subtraction. While Ms. Santi was out of school for a few days, the substitute teacher had taught her students to subtract when the problem asked, "How many more?" When Ms. Santi returns to school, she decides to check her students' understanding of this problem type and poses the following problem:

> Robert had 14 lollipops. Jeffrey had only 6 lollipops. How many more lollipops did Robert have?

The students quickly get to work, using a variety of manipulatives to solve the problem. While they work, Ms. Santi walks around the room and talks with them about their strategies. To her amazement, many of her students have immediately added the numbers and have 20 as their answer. Then Robert speaks up. "I drew all the lollipops too, but I only counted eight. I know the answer is eight." His work, however, shows a row of 6 lollipops at the top and 14 more scattered below. Melissa notes she got 8 too. "I made the numbers with color tiles, and I put them beside each other, and I didn't count the ones that match. I counted eight." Lourdes, looking on, insists, "You have to count all of them, because you have more than eight." At this point, most of the class begins to discuss whether you need to count all the lollipops or just some of them. Then Corey shouts excitedly, "I know how to show it! I made it with snap cubes, and I know it's eight because you put Brandon's and Jeffrey's side by side and you just count the extras!" Corey holds up his snap-cube graph for the class to see.

Figure 1.1



Immediate comprehension sweeps the room and several students repeat, "You just count the extras."

Trying to help her students make a connection with the previous textbook subtraction problem, Ms. Santi then asks if they thought subtraction could solve the lollipop problem. No one replies. She then asks them about the textbook subtraction problems they had completed successfully the week before. Marjorie says, "The teacher said we were to subtract on those problems." The other students quickly agree. Ms. Santi writes

$$\begin{array}{r} 14 \\ -6 \\ \hline 8 \end{array}$$

on the board and asks the students how this differs from Corey's snap-cube solution. Lourdes says, "The numbers are the same, but you have a minus sign." There is silence and then Robert raises his hand. "When you take-away, aren't you matching the six with the six in the fourteen? Then the eight are just the extras."

There is a substantial difference between knowing how to do a mathematics procedure and knowing when to do a mathematics procedure. To further develop her students' understanding of the meanings of addition and subtraction, Ms. Santi has provided a problem context that encourages them to make sense of their use of mathematics.

Precautions and Possible Pitfalls

It is not uncommon for children, if given the opportunity to make sense of the mathematics they are doing, to solve the problem Ms. Santi posed by counting on—that is, beginning with six objects and counting eight more to make fourteen—or, as Corey said, by "counting the extras." However, many teachers—due to their own prior mathematics experiences—tend to ignore the "more" in the problem and obtain a solution through subtraction. Teachers need to build on students' mathematics intuitions by carefully bridging these solution techniques. While Corey's solution is a practical technique for small differences, subtraction, if properly understood, may be more practical for comparison of larger numbers.

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STRATEGY 3: Encourage young children's fluent computation.

NCTM Standard

Compute fluently and make reasonable estimates.

What Research and the NCTM Standards Say (NCTM, 2000)

Young children often initially compute by using objects and counting. However, prekindergarten through Grade 2 teachers need to encourage them to shift, over time, to solving many computation problems mentally or with paper and pencil so as to record their thinking. Students should develop strategies for knowing basic number combinations (the single-digit addition pairs and their counterparts for subtraction) that build on their thinking about, and understanding of, numbers. Fluency—that is, students are able to compute efficiently and accurately with single-digit numbers—with basic addition and subtraction number combinations is a goal for the preK–2 years.

As students work with larger numbers, their strategies for computing play an important role in linking less formal knowledge with more sophisticated mathematical thinking. Research provides evidence that students will rely on their own computational strategies (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). Such inventions contribute to their mathematical development (Gravemeijer & Eskelhoff 1994; Steffe 1994).

Classroom Applications

In this vignette (adapted from NCTM, 2000, pp. 86–87), we listen as Mr. Daley's second graders are being encouraged, building on classroom discussions of addition, to model and record their addition strategies.

Mr. Daley has posed the following addition problem to his class of second graders:

> We have 153 students at our school. There are 273 students at the school down the street. How many students are in both schools?

While his students work, he walks around the room observing and listening to their strategies. His students give a variety of responses that illustrate a range of understandings. For example, Raul models the problem with bean sticks that the class has made earlier in the year, using hundreds rafts, tens sticks, and loose beans. He then draws a picture of his model and labels the parts, "3 rafts," "12 tens," "6 beans."

Ana first adds the hundreds and records 300 as an intermediate result. She then adds the tens, and, keeping the answer in her head, adds the ones. Finally, she adds the partial results—300 + 12 tens + 6—and writes down 426 as the answer. Other students use the conventional algorithm (stacking the addends and then adding the ones, adding the tens and renaming them as hundreds and tens, and finally adding the hundreds). Most do this accurately, but some write 3126 as their answer. Stacy finds the answer using mental computation and writes nothing down except her answer. When Mr. Daly asks her to explain, she says, "Well, two hundreds and one hundred are three hundreds, and five tens and five tens are ten tens, or another hundred, so that's four hundreds. There's still two tens left over, and three and three is six, so it's four hundred and twenty-six."

Problems such as the one posed by Mr. Daley give his students meaningful opportunities to reflect on and fluently apply their knowledge of numbers and operations. Simultaneously, he gains deeper insights into his students' misconceptions and mistakes and thus is better able to design, if necessary, further and relevant practice. Later in the lesson, Mr. Daley will ask his students to explain their strategies to their classmates. Such presentations can provide further opportunities for practice, application, and reflection.

Precautions and Possible Pitfalls

Properly understood, facility in computation is neither purely a product of practice nor purely a product of understanding growing, as it does, out of their interplay. By allowing students to work in ways that have meaning for them and by encouraging them to develop efficient strategies, teachers can gain insight—through student explanations—into students' developing understanding and give them guidance. Such meaningful practice, rather than only drilling isolated number facts, is necessary to develop fluency with basic number combinations and strategies with multidigit numbers.

Sources

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Grades 3–5

Most students enter Grade 3 with enthusiasm for, and interest in, learning mathematics. In fact, nearly three-quarters of U.S. fourth graders report liking mathematics (Silver, Strutchens, & Zawojewski, 1997). They find it practical and believe that what they are learning is important. If the mathematics studied in Grades 3–5 is interesting and understandable, the increasingly sophisticated mathematical ideas at this level can maintain students' engagement and enthusiasm. But if their learning becomes a process of simply mimicking and memorizing, they can soon begin to lose interest. Instruction at this level must be active and intellectually stimulating and must help students make sense of mathematics (NCTM, 2000).

STRATEGY 4: Encourage an understanding of the structure of numbers and relationships among numbers.

NCTM Standard

Understand numbers, ways of representing numbers, relationships among numbers, and number systems.

What Research and the NCTM Standards Say (NCTM, 2000)

In Grades 3–5, students' study and use of numbers should be extended to include larger numbers, fractions, and decimals. They need to develop strategies for judging the relative sizes of numbers. They should understand more deeply the multiplicative nature of the number system, including the structure of 786 as 700 plus 80 plus 6. Students who understand the structure of numbers and the relationships among numbers can work with them flexibly. They recognize and can generate equivalent representations for the same number. For example, 36 can be thought of as $30 + 6$, $20 + 16$, 9×4, $40 - 4$, three dozen, or the square of 6. Each form is useful for a particular situation. Thinking of 36 as $30 + 6$ may be useful when multiplying by 36, whereas thinking of it as 6 sixes or 4 nines is helpful when considering equal shares. Students who understand the structure of numbers and the relationships among numbers can work with them more flexibly (Fuson, 1992). Students need to have many experiences decomposing and composing numbers in order to develop such understanding.

Classroom Applications

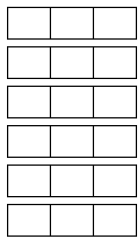
In this vignette (adapted from Fosnot & Dolk, 2001, p. 45), we listen as Ms. Freeman's third graders study, within a particular context, the associative and commutative properties of multiplication.

Ms. Freeman has asked her students to investigate how many different-sized boxes they can make that would hold 36 chocolates (for example, a box of $2 \times 6 \times 3$ chocolates).

As Ms. Freeman walks around the room listening to her students discuss the problem, she stops at the table where Sara and Mercedes are working. Sara and Mercedes have 36 multilink cubes (each cube, measuring a cubic unit, represents a chocolate) arranged in a box with dimensions of 2 by 3 by 6. On graph paper they have drawn the base of the box as 3 by 6. Next to the drawing Mercedes has written, "(3 × 6) × 2. The bottom has 3 rows of 6. It has 2 layers."

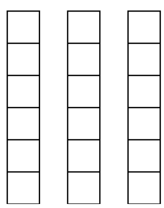
Sara smiles at Ms. Freeman and then turns to Mercedes. "I'm confused," she explains, "I see six rows of three."

Figure 1.2



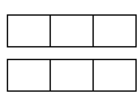
"Where?" Mercedes asks. Sara points. "See, three, six, nine, twelve, fifteen, eighteen." "Oh, yeah," Mercedes exclaims, "but see, six and six and six."

Figure 1.3



"Hey, let's write both!" They write, "(3 × 6) × 2" and "(6 × 3) × 2." Then Mercedes asks, "How else can we turn it? Oh, I know, we could make this side be the bottom!" She turns the box so that the 3-by-2 side becomes the base.

Figure 1.4



"Now we have one, two, three, four, five, six layers!" "Yeah, and we can write it two ways," Sara, looking at Ms. Freeman, proudly declares as she writes, "(2 ×3) ×6" and then "(3 × 2) × 6."

Ms. Freeman has provided a mathematics context in which her students are being encouraged to explore the various multiplicative representations of 36. Even though Sara and Mercedes are presently working with only the factors 2, 3, and 6, they are developing a practical awareness of the important mathematics principles of commutativity and associativity. Experience with these principles is essential if students are to deepen their understandings of number representations and relationships.

Precautions and Possible Pitfalls

Manipulatives can provide opportunities for students to model and investigate significant number relationships such as the associative and commutative properties. However, manipulatives, unless purposively and deliberately utilized for the doing of important mathematics, are often ineffective in developing students' mathematical knowledge or fluency. A child playing with square blocks, for example, is unlikely to naturally realize that the cube of the length of a side is the volume. Manipulatives do not transparently embody mathematical truths (Ball, 1992).

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STRATEGY 5: Encourage an understanding of the meanings of multiplication and division and how they relate to each other.

NCTM Standard

Understand meanings of operations and how they relate to one another.

What Research and the NCTM Standards Say (NCTM, 2000)

In Grades 3–5, students should focus on the meanings of, and relationship between, multiplication and division. It is important that students understand what each number in a multiplication or division expression represents. For example, in multiplication, unlike addition, the factors in the problem can refer to different units. Modeling multiplication problems with pictures, diagrams, or concrete materials helps students learn what the factors and their product represent in various contexts.

Students can extend their understanding of multiplication and division as they consider the inverse relationship between the two operations. Using models—such as multiplicative arrays (Fuson, 2003) or calculators (NCTM, 2000)—to explore the effect of multiplying or dividing numbers can also lead to a deeper understanding of these operations.

Classroom Applications

Ms. Pierce's fourth-grade class has recently completed a unit on multiplication and division. In this vignette (adapted from NCTM, 1991), we listen as Ms. Pierce introduces the topic of factors and multiples. Ms. Pierce has chosen to use the calculator as a tool in this initial lesson.

The fourth graders, using the automatic constant feature of their calculators (for example, pressing 5 + = = = . . . yields 5, 10, 15, 20 . . . on the calculator display), generate lists of the multiples of different numbers. They also use the calculator to explore the factors of different numbers. To encourage the students to deepen their understanding of the structure of numbers, Ms. Pierce urges them to look for patterns and to make conjectures by asking them, "Do you see any patterns in the lists you are making? Can you make any guesses about any of those patterns?"

Shannaz and Maria raise a question that has attracted the interest of the whole class:

Are there more multiples of 3 or more multiples of 8?

"What do the rest of you think?" Ms. Pierce asks. "How could you investigate this question? I want you to work on this a bit on your own or with a partner, and then let's discuss what you come up with."

The students pursue the question excitedly. The calculators are useful once more as students generate lists of the multiples of 3 and the multiples of 8. Groups form around particular arguments. Sinting, Sharda, and

Jonnatun argue that there are more multiples of 3 because in the interval between 0 and 20, there are more multiples of 3 than multiples of 8. Raul, Melissa, and Leigh are convinced that the multiples of 3 are "just as many as the multiples of eight because they go on forever." John and Derrick, thinking that there should be more multiples of 8 because 8 is greater than 3, form a new conjecture about numbers—that the larger the number, the more factors it has.

Ms. Pierce takes up this last conjecture and says to the class, "That's an interesting conjecture. Let's just think about it for a second. How many factors does, say, three have?" Michel replies, "One and three." "Let's try another one," continues Ms. Pierce. "What about twenty?'" Natasha replies, "One and twenty, two and ten, four and five." The period is drawing to an end, and as Ms. Pierce looks up at the clock, Brody asks, "But what about seventeen? It doesn't seem to work?" Ms. Pierce smiles and replies, "That's one of the things that you could examine for tomorrow. I want all of you to see if you can find out if this conjecture always holds."

Ms. Pierce has provided a context that provides an opportunity for her students to build on their knowledge of multiplication and division in order to practice the mathematical notions of factor and multiple and to explore how they are related to each other. Explorations such as this are also important in enriching students' understanding and sense of the multiplicative structure of numbers.

Precautions and Possible Pitfalls

Students can learn mathematics more deeply with the appropriate use of technology (Groves & Stacey, 1998). Technology should not, however, be used as a replacement for basic understandings and intuitions; rather, it can and should be used to help foster those understandings and intuitions. Students at this age should begin to develop good decision-making habits about when it is useful and appropriate to use other computational methods, rather than reach for a calculator. Teachers should create opportunities for these decisions as well as make judgments about when and how calculators can be used to support learning.

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STRATEGY 6: Encourage students to compute fluently and make reasonable estimates.

NCTM Standard

Compute fluently and make reasonable estimates.

What Research and the NCTM Standards Say (NCTM, 2000)

Research suggests that by solving problems that require calculation, students develop methods for computing and also learn more about operations and properties (McClain, Cobb, & Bowers, 1998; Schifter, 1999). As students develop methods to solve multidigit computation problems, they should be encouraged to record and share their methods. As they do so, they can learn from one another, analyze the efficiency and generalizability of various approaches, and try one another's methods. In the past, common school practice has been to present a single algorithm for each operation. However, more than one efficient and accurate computational algorithm exists for each arithmetic operation. Further, if given the opportunity, students naturally invent methods to compute that make sense to them (Fuson, 2003; Madell, 1985).

Classroom Applications

In this vignette (adapted from NCTM, 2000, pp. 153–154), we listen as Ms. Sparks gives her fifth graders the opportunity to share their computational procedures for division.

Ms. Sparks has asked her students to share their solutions to a homework problem, 728 ÷ 34. She has asked several students to put their work on the board to be discussed, and she has deliberately chosen students who had approached the problem in several different ways. As the students put their work on the board, Ms. Sparks circulates among the other students, checking their homework.

Henry has written his solution:

$$34 \times 10 = 340$$
$$34 \times 20 = 680$$

$$\begin{array}{r} 680 \\ +\ 34 \\ \hline 714 \end{array} \qquad \begin{array}{r} 728 \\ -714 \\ \hline 14 \end{array}$$

Henry explains to the class, "Twenty thirty-fours plus one more is twenty-one. I knew I was pretty close. I didn't think I could add any more thirty-fours, so I subtracted seven hundred fourteen from seven hundred twenty-eight and got fourteen. Then I had twenty-one remainder fourteen." Students nod their heads in agreement.

Michaela shows her solution:

$$\begin{array}{r} 21 \\ 34\overline{)728} \\ \underline{68} \\ 48 \\ \underline{34} \\ 14 \end{array}$$

and says, "Thirty-four goes into seventy-two two times and that's sixty-eight. You gotta minus that, bring down the eight, then thirty-four goes into forty-eight one time." Some children do not understand Michaela's explanation, and Ms. Sparks asks if anyone can see parts of Michaela's and Henry's work that are similar.

Fashen says hesitantly, "Well, there is a six hundred eighty in Henry's and a sixty-eight in Michaela's." Ms. Sparks asks Michaela about the 68 and she replies that it is 2 times 34. Ms. Sparks says, "So, I don't get what you're saying about two times thirty-four. What does this two up here in the twenty-one represent?" Samir says, "It's twenty," and Henry remarks, "But twenty times thirty-four is six hundred eighty, not sixty-eight." Ms. Sparks writes a zero after the 68 and says, "So what if I wrote a zero here to show that this is six hundred eighty? Does that help you see any more similarities?" Maya says, "They both did twenty thirty-fours first," and Rita exclaims, "I get it. Then Michaela did, like, how many more are left, and it was forty-eight, and then she could do one more thirty-four."

Regardless of the particular algorithm used, students exhibit *computational fluency* when they are able to clearly explain their particular method, use it to compute accurately and efficiently, and recognize that other methods exist (NCTM, 2000). However, students need help in doing such work. Ms. Sparks saw important mathematics relationships between the methods described by Henry and Michaela, but she doubted that any of her students would initially see these relationships. Through deliberate questioning, she was able to help students focus on the ways in which both Michaela's and Henry's methods used multiplication to find the total number of 34s in 728 and helped students clarify what quantities were represented by the notation in Michaela's solution.

Precautions and Possible Pitfalls

Procedural and conceptual describe ways of doing mathematics and should not be used for classifying mathematics solutions. Although Michaela's solution uses the standard long division algorithm and Henry's solution seems more of his own invention (it is, in fact, common in the elementary grades), both are valid mathematics solutions. Ms. Sparks's deliberate questioning helps give Michaela's solution a conceptual basis and highlights its efficiency. However, Henry's solution can be equally or more efficient when, for example, one divides 704 by 10. For such a division, a *computationally fluent* student would not use the standard long division algorithm.

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