1

Writing and Mathematics

An Introduction

"Would you tell me please, which way I ought to go from here?" asked Alice.

"That depends a good deal on where you want to get to," said the Cat.

—Alice's Adventures in Wonderland
(Carroll, 1965a, p. 71)

Instructional programs from prekindergarten through grade 12 should enable all students to

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely. (National Council of Teachers of Mathematics [NCTM], 2000, p. 159)

> Imagine a classroom, a school, or a school district where all the students have access to high-quality, engaging mathematics instruction. . . . Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally *and in writing* [italics added], students communicate their ideas and results effectively. (NCTM, 2000, p. 3)

THE PURPOSE OF THIS BOOK

The primary purpose of *Write for Mathematics* is to help you *teach* your students how to communicate in mathematics and with mathematics so that they develop the deepest possible understanding and application of this essential subject.

We have written this book for elementary and secondary school teachers who are being asked to integrate what were once assumed to be two different and even disparate subjects—mathematics and writing—and who may find themselves unprepared for this daunting task. We focus primarily on teaching students how to write for mathematics and about mathematics, an idea that has captured educators' interest over the past 20 years or so (Meier & Rishel, 1998) and is now included as part of mathematics assessment throughout the nation (NCTM, 2004).

WHY WRITE IN MATHEMATICS?

The benefits of linking writing with mathematics are cited in numerous books and periodicals. McLoughlin (1998) explains that teachers who ask their students to write about mathematics are able to

- gain insight into their students' mathematical thinking,
- diagnose their students' misconceptions,
- assess students' study habits and attitudes, and
- evaluate their own teaching techniques.

Among the principles for school mathematics cited in *Principles and Standards for School Mathematics* (NCTM, 2000) are learning and assessment. Learning means "understanding" and assessment must "furnish useful information to both teachers and students" (p. 11). It is through writing, as well as through algorithms, that students can gain and show their understanding and provide others with "useful information."

Yet many teachers worry that if they teach writing during mathematics time, they may take away instructional time from mathematics itself. Teachers of mathematics worry about "correcting" writing and spending time going over students' papers. They are concerned (understandably) about taking the role of the language arts or English teachers and thinking about grammar and punctuation and format. However, research by Gopen and Smith (1990) indicates that teachers can cover "the [mathematics] material" and "incorporate writing assignments . . . with significant success and without unduly burdensome effort" (p. 18). This research has also been corroborated by Reeves (2002) and Burns (1992). However, teachers also need to have an understanding of the tools or strategies of writing (see Rothstein & Lauber, 2000). Mathematics, after all, is a written language and mathematicians write about mathematics. Miller (1991) focuses on the advantages of teaming mathematics with writing and points out that "because writing leads people to think, improved mastery of mathematics concepts and skills is possible if students are asked to write about their understanding" (p. 520). Writing about a topic requires students to internalize important concepts as well as analyze, compare facts, and synthesize information (Kennedy, 1980). Studies by Freeman and Murphy (1992), Johnson (1983), and Rishel (1991, 1993) indicate that the integration of writing with mathematics fosters greater student interest and higher student achievement levels in mathematics.

By systematically merging writing with mathematics in every grade, students learn to think mathematically as well as express their mathematical ideas and concepts. Now that schools have declared that writing and mathematics are a "team," teachers need to have a wide range of writing strategies that are *specifically suited* to mathematics and that clearly integrate what needs to be taught in mathematics with the writing activities. We believe that this book will provide you with many of these strategies.

Concerns of Mathematics Teachers

Only a few years ago, school administrators felt they had to apologize to mathematics teachers who were required to attend writing workshops. Today, however, "students at all levels are being asked to write in their mathematics classes" and future mathematics teachers studying in universities across the country are required to pass writing assignments that are over 400 words in length (Brewster, Fleron, Giuliano, Hoagland, & Rothermel, 2000). Less than a generation ago, writing (and its handmaidens—spelling, grammar, word usage, and punctuation) was the domain of English or language arts teachers and possibly a minor concern of social studies teachers. Mathematics teachers (or those who taught mathematics within the total curriculum) had enough to do when teaching students to solve word problems without asking them to construct word problems. These teachers asked, How important is it for students to write about mathematics if they have difficulty understanding the basics of mathematics? Even those mathematics teachers who recognized the value of writing in mathematics asked, Is it my responsibility to correct, grade, and comment on my students' compositions? Above all, with so much difficult material to cover and with so much drill and skill required, how much time could—or should—be set aside for writing in mathematics? How is it possible for teachers to meet state curriculum standards that require students to "become mathematically confident by communicating and reasoning mathematically, by applying mathematics in real world settings, by solving problems through the integrated study of number systems, geometry,

algebra, data analysis, probability, and trigonometry" or to "apply technological knowledge and skills to design, construct, use, and evaluate products and systems to satisfy human and environmental needs" (Barker et al., 1996, p. 1)?

Standards such as those listed above seem overwhelming, especially when mathematics is only one part of an already demanding curriculum. Even more overwhelming is the requirement that students must now "write" for mathematics with clarity and comprehension, using appropriate syntax and written conventions. This demand is an extra heavy burden since most teachers—both elementary and secondary—have seen themselves as either teaching mathematics or teaching language. Rarely can teachers recall their English teachers explaining how to write statements in algebra or geometry or their mathematics teachers helping them write a comparative essay on the mathematical efficacy of Roman numerals versus Arabic numerals.

THINKING AND MATHEMATICS

No one doubts that mathematics requires thinking, yet the relationship of thinking to mathematics is not always spelled out either to the teacher or the students. Strong, Silver, and Perini (2001) denote five key areas of thinking: knowledge acquisition, inquiry, problem solving, communication, and reflection. They then apply these areas to mathematics. Figure 1.1 shows the five key areas applied to a student learning multiplication.

Hirsch (1996) emphasizes the reciprocal relationship between automatic mastery of repeated lower-level activities and higher-order thinking skills mediated by "particular vocabulary meanings" (p. 151). In mathematics, "the repeated grammar-like operations [are] rules, and the more vocabulary-like, content-area [are] schemas" (p. 151). According to Hirsch, "the development of math skills with the . . . development of communication/learning skills suggests . . . a general structure for real-world problem solving and critical thinking skills" (p. 151). Combining writing with mathematics is a natural partnership for achieving the high-level standards of learning and thinking that schools are seeking.

MATHEMATICS AND STANDARDS

As states began implementing performance standards in all subjects, they looked more closely at what students need to know and be able to do in a complex, technological society. In *Promises to Keep* (Wurtz & Malcolm, 1993) and *Performance Standards* (National Center on Education and the Economy [NCEE], 1995), performance standards are described as a "social compact" with students. This compact promises students a quality education and the means to apply what they have learned in school to their careers and other aspects of their lives. In mathematics, this application (termed "applied learning" in the above documents) focuses on "helping students to be productive members of society, as individuals who apply the knowledge gained in school . . . to analyze problems and propose solutions, to *communicate effectively* [italics added] and coordinate action with others, and to use the tools of the information age workplace" (Rizzo et al., 1998, p. 5). In *Principles and Standards,* the first principle is "equity," which is defined as "high expectations and strong support for all students" (NCTM, 2000, p. 11).

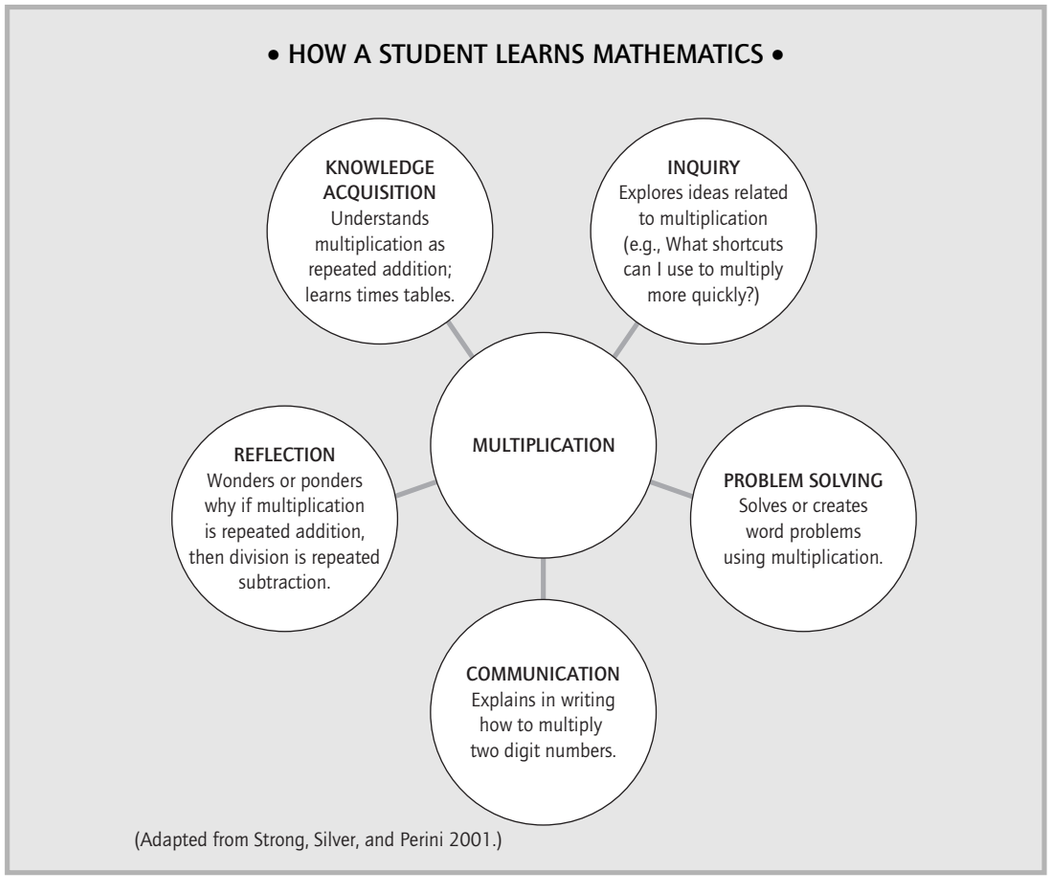

Figure 1.1

The National Council of Teachers of Mathematics (NCTM, 2000), the leading educational organization representing mathematics teachers, also explains the societal needs for mathematics:

- mathematical literacy for everyday life,
- cultural literacy (understanding the historical/cultural significance of mathematics),
- mathematical literacy for the workplace, and
- mathematical literacy for scientific and technical careers (pp. 3–4).

In addition, the NCTM (2000) makes the following comments that relate to writing in mathematics:

- As students learn mathematics, they will develop an increasing repertoire of problem-solving skills, a wide range of mathematical "habits of the mind," and increasing sophistication in mathematical argument.
- Also students should become proficient in expressing themselves mathematically, both orally and in writing, gaining fluency in the language of mathematics and being able to make connections within mathematics and from mathematics to other disciplines. (p. 1)

The NCTM (2000) created 10 standards for mathematics instruction. The standards are divided into two categories: content standards and process standards. In organizing *Write for Mathematics*, we drew particularly from the eighth standard because this standard requires mathematics instructional programs to use communication to foster a deeper understanding of mathematics. This standard states that students from kindergarten through grade 12 should be able to

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely. (p. 59)

Write for Mathematics not only focuses on the eighth standard, but it also addresses the other NCTM standards within the context of writing for mathematics by organizing these standards into 10 strategies that incorporate a variety of writing genres (see Figure 1.2: Planning Wheel). The first seven strategies—Taxonomies; Composing With Keywords; Metacognition; Defining Format; Morphology and Etymology; Profiles and Frames; and Reasons, Procedures, Results—specifically target the NCTM standards. The first four strategies focus largely on terminology and vocabulary; the Profiles and Frames and Reasons, Procedures, Results focus on different types or genres of mathematical writing products. The last three strategies—Who's Who in Mathematics, Where in the World, and Personifications and Interactions—guide students in relating mathematics to everyday life, students' cultural heritage, the workplace, and the scientific and technical community (cited in NCTM, 2000, p. 4). When these strategies are systematically taught and practiced, students will become mathematically literate. Chapters 2 through 11 address and expand upon each of these 10 strategies.

WRITING STRATEGIES RELATED TO THE TEACHING OF MATHEMATICS

In our earlier book, *Writing as Learning* (Rothstein & Lauber, 2000), we explained that learning to write requires fluency plus organization. Simply stated, a writer must know the language of the subject or topic and must know the different types of organizational schema (also called genres) that are appropriate for specific subject-area writing. In social studies, for example, students studying the pilgrims need to know the vocabulary related to the pilgrims: England, religious freedom, colony, compact, native population, and so forth. If students do not understand this specialized vocabulary, they will view the story of the pilgrims as merely an American legend repeated each fall. To further understand the story, the students must express their knowledge in a variety of writing forms: explanation, narrative, biography, personal, poetic, and more. In mathematics, "developing fluency requires a balance and connection between conceptual understanding and computational proficiency" (NCTM, 2000, p. 35). The organization schema in mathematics relates to representational ways that computation understanding is presented.

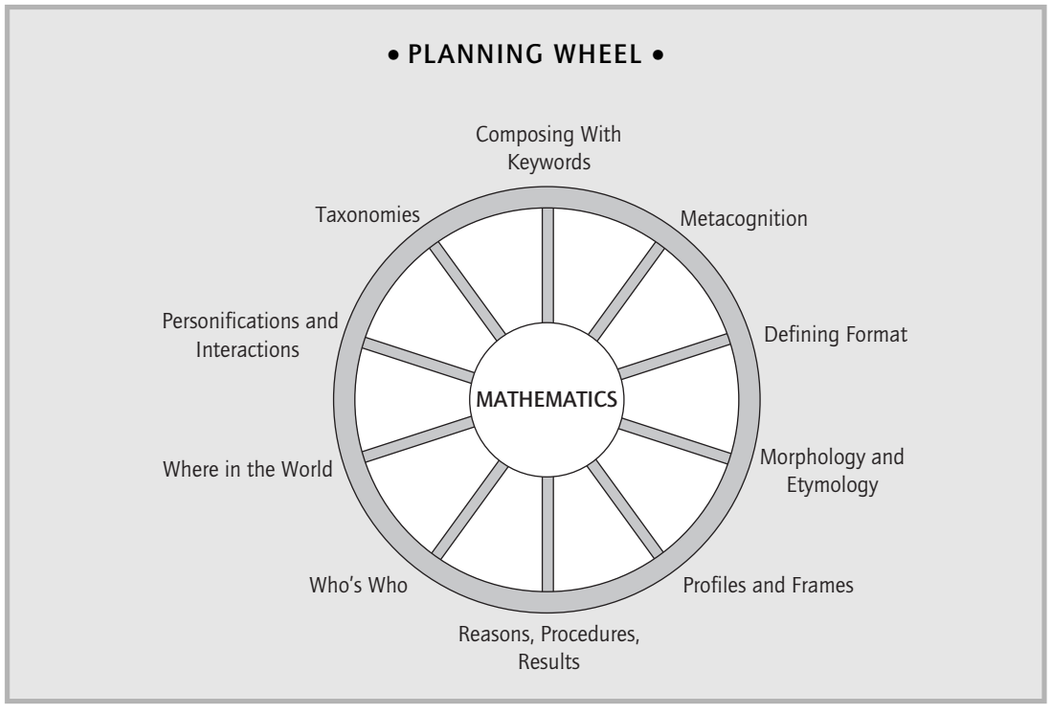

Figure 1.2

The concept of fluency-plus-organization in mathematics can be more readily understood through the writings of Howard Gardner who, in his work on multiple intelligences, defines intelligence as "the ability to solve problems or fashion products that are of consequence in a particular cultural setting or community" (1993, p. 15). Gardner speaks of intelligences (plural) rather than intelligence (singular) to denote problem-solving ability in specific domains, such as verbal/linguistic, musical/rhythmic, logical/mathematical, and others. He stresses the importance of focusing on both "the content of instruction and the means or medium for communicating that content" (p. 32). The content of mathematical instruction is well known; all of us remember our years of "math"—memorizing multiplication tables, measuring with rulers and yardsticks, drawing graphs, constructing triangles, and taking weekly math tests so that we could get a grade for the report card. What we may not remember (or didn't learn) are the different means of communicating mathematical content other than through tests or occasional projects.

The 10 writing strategies (Figure 1.2), when implemented systemically from primary grades through high school in every subject area, including mathematics, teach students not merely the content but the means of communicating that content. Each of these strategies has a place on the Planning Wheel and is a way of visualizing the order of the strategies and their relationship to each other. Since this book is about writing for mathematics, MATHEMATICS is placed in the center of the wheel with the idea that content will be delivered through these various strategies. Figure 1.3 defines each strategy and shows related learning extensions and applications for each strategy.

STRATEGIES FOR WRITING AND COMMUNICATING MATHEMATICS

Strategy	Learning Extension	Applications
Taxonomies: Alphabetical lists of terms related to mathematical topics (e.g., algebra, numbers, measurement)	**Taxonomies** are for • building fluency in mathematical vocabulary; • taking notes in mathematics and keeping track of terminology; • assessing pre- and post-mathematical knowledge.	**Taxonomies** extend knowledge of specific mathematical vocabulary related to • number sense and operations; • patterns, functions, and algebra; • geometry and spatial sense; • measurement; • data analysis, statistics, and probability; • concepts in geography, money, science, and other subjects.
Composing With Keywords: Using the words from the Taxonomies to compose mathematically related sentences and paragraphs	**Composing With Keywords** is for • composing sentences using mathematical terms; • writing paragraphs for mathematics; • focusing on a mathematical topic or concept.	**Composing With Keywords** develops ability to compose mathematical ideas in • response logs; • journals; • proofs.
Metacognition: Self-awareness of mathematical knowledge (e.g., using statements such as, I know that I know something about a polygon)	**Metacognition** guides students in • questioning mathematical ideas; • searching and researching for mathematics around us; • responding in writing to prior and new mathematics-related knowledge.	**Metacognition** enhances questioning skills: • How is what I already know in mathematics connected to what I need to know? • What else must I know in mathematics? • How can I best learn what I need to know mathematically?
Defining Format: A three-part format to define a mathematical term (e.g., number) that consists of the question, the category, and the characteristics	**Defining Format** provides a template for • asking questions related to mathematical terms; • categorizing mathematical terms; • adding details to expand mathematical knowledge; • writing clear mathematical statements.	**Defining Format** promotes learning how to • write accurate mathematical definitions; • categorize mathematical terms accurately; • compare mathematical concepts; • support mathematical ideas; • state mathematical facts.

Figure 1.3

STRATEGIES FOR WRITING AND COMMUNICATING MATHEMATICS

Strategy	Learning Extension	Applications
Morphology and Etymology: The study of the formation and history of words related to mathematics (e.g., number, numeral,numerical, enumerate, numeration, numeracy)	**Morphology and Etymology** focuses on • understanding the meaning of related mathematical terms; • clarifying mathematical terms that have multiple meanings; • learning mathematical idioms.	**Morphology and Etymology** helps students • create mathematical word stories; • relate mathematics to the world of mythology and legends; • write mathematical alliterations, poetry, puns, jokes, and do other mathematical word play.
Profiles and Frames: Templates for outlining information in all mathematical subcategories (e.g., arithmetic, geometry, algebra)	**Frames and Profiles** provide additional templates for • organizing research of mathematical topics; • paraphrasing and restating mathematical ideas and concepts.	**Frames and Profiles** help students • make oral presentations; • report on mathematics-related topics; • create mathematical reports.
Reasons, Procedures, Results: An organizational essay format for detailing reasons, procedures, and results related to mathematical ideas and concepts	**Reasons, Procedures, Results** provide organizational essay formats for • focusing on the mathematical topic; • adding supporting details; • writing paragraphs or reports.	**Reasons, Procedures, Results** are for • solving mathematical problems; • explaining mathematical ideas; • persuading others of mathematical concepts or beliefs; • reasoning, proving, and refuting mathematical statements; • creating personal essays related to mathematics.
Who's Who in Mathematics: Formats for researching and writing about accomplished mathematicians and their contributions to mathematics	**Who's Who in Mathematics** is for • biographic research related to mathematicians; • relating mathematics to other subject areas; • learning advanced mathematical ideas developed by renowned mathematicians.	**Who's Who in Mathematics** helps students in • making connections to the mathematical community; • understanding mathematical minds; • considering careers that use mathematics or need mathematicians.

(Continued on page 10)

STRATEGIES FOR WRITING AND COMMUNICATING MATHEMATICS

Strategy	Learning Extension	Applications
Where in the World: Strategies for relating mathematics and mathematical ideas to geography, culture, and mathematical ideas (e.g., Greeks, Arabs, Mayans, maps)	**Where in the World** shows the • relationship of geography to mathematics; • mathematical aspects of charts, graphs, maps, globes; • related vocabulary of mathematics and geography.	**Where in the World** builds awareness of • social issues and mathematics; • exploration and mathematics; • demographics and mathematics; • history of mathematics.
Personifications and Interactions: Strategy in which the writer assumes the persona of a mathematical term or concept and writes to another mathematical persona (e.g., a square writing to a circle)	**Personifications and Interactions** is another way to • write mathematical research; • direct writing to an audience; • express mathematical ideas in an informal or imaginative voice.	**Personifications and Interactions** • add humor to mathematics; • bring reality and fantasy to mathematical ideas; • foster in-depth understanding of concepts by assuming the persona of the concept.

Figure 1.3 *(Continued from page 9)*

By combining the 10 writing strategies illustrated in the Planning Wheel (Figure 1.2) with the 10 NCTM standards, you will have an expanded way of teaching students how to learn mathematics, how to think mathematically, and how to appreciate the remarkable invention of mathematics given to all humanity by teachers, philosophers, traders, and many others who needed to find a way to solve problems and fashion products.

HOW TO USE THIS BOOK FOR WRITING IN MATHEMATICS

The purpose of this book is to give you the tools that will help make your students mathematically literate. By juxtaposing mathematics with literacy, you will focus on uniting two intelligences—logical/mathematical with verbal/linguistic—a powerful partnership that Gardner (1993) points out is the principal basis of IQ and SAT tests (p. 20). The book is arranged so that you can develop many of your mathematical lessons within the framework of this partnership regardless of what skill or topic you are presenting. No matter which grade you teach, you will be able to enhance your students' knowledge and communication ability by incorporating the writing strategies in this book with your mathematical instruction.

Introduce each strategy slowly and thoroughly. Keep in mind that one strategy builds upon the other, no differently than in traditional mathematics instruction. Begin by teaching the strategies consecutively and *within the context of the mathematics you are teaching*. The activities presented cover a wide range of topics and grade levels as exemplars, so they will often have to be adapted to your own students' levels and abilities. However, every strategy can be effectively used with all students.

When students have mastered the strategies, you may use them in any order to maximize learning and understanding. For example, in the middle of the year your students might use Defining Format, add to their Taxonomy, create a Profile, and end with a Personification. You will find that the strategies will guide your students in applying their knowledge; they will gain proficiency in solving word problems, state reasons with proof, make connections between two mathematical ideas or concepts, and explain clearly and cogently their mathematical representations.

Chapters 2 through 11 each contain a section called **Mathematics the Write Way.** You can use the activities in this section as they are presented, or you may modify them to meet your students' needs or your district's curriculum. In every chapter, the sample activities focus on six mathematical areas: number and operations, algebra, geometry, measurement, data and analysis, and money. The first five areas are derived from the NCTM (2000) content standards. We added the sixth content area—money—because it has its own specialized vocabulary and approach.

Chapters 2 through 11 also include a section called **Internet Links.** This section will help you find activities and ideas for expanding your students' mathematical knowledge. These activities and ideas can be integrated with the specific strategy students are learning.

An issue for many teachers is time. How much time to spend on each strategy is a reasonable and essential question. One answer comes from *Principles and Standards,* which states that "school mathematics programs should not address every topic every year" and that "instructional programs . . . focus on important mathematical areas" (NCTM, 2000, p. 7). We would like to add that building mathematical understanding and communication does take time and, when given the time, results in the enhancement of the intelligence that allows us, as Gardner (1993) so cogently says, to solve problems and fashion products.

Below is a summary of each chapter that describes the writing strategy and its relationship to specific aspects of mathematics. Notice that every strategy is listed with a slogan that focuses on the major idea or purpose of that strategy, followed by the strategy name, as in "Words Are Free!—Taxonomies for Mathematics." Share these slogans with the students and post them in your classrooms. (If you would like to see how all the strategies fit together after you have taught them, go to Chapter 12.)

Chapter 2: **Words Are Free!—Taxonomies for Mathematics** introduces the concept of Taxonomies—organized lists of words pertaining to mathematical areas and topics for the purpose of building fluency and a personal thesaurus. These Taxonomies include words that refer to the following:

- number and operations (e.g., odd, digit, positive)
- algebra (e.g., same, similar, equivalent, attributes)
- geometry (e.g., scalene, congruent, polyhedron)
- measurement (e.g., metric, weight, time)
- data analysis and probability (e.g., sample, predict, compare, analyze)
- problem solving with money (e.g., penny, quarter, value, worth)

Chapter 3: **Have Words, Can Write—Composing With Keywords** shows the students how to use the mathematical words from Taxonomies to create sentences that state mathematical ideas in all strands of the mathematics curriculum. Composing with keywords provides students with ongoing, continuous practice in writing for mathematics.

Chapter 4: **Know What You Know—Metacognition for Mathematical Thinking** gives the students templates for expressing their prior knowledge and their developing knowledge as they learn more complex mathematical concepts. Metacognition statements help students "put in writing" what they know or think about a mathematical term or concept (e.g., triangle) before they receive instruction and what they know or think after they receive instruction.

Chapter 5: **Write to a Martian—Defining Format** brings the words from the Taxonomies into full focus and requires the writer (student) to explain the word (or concept) to a distant audience, a.k.a. The Martian. By focusing on a distant or unknowing audience, the student must clearly define significant mathematical terms (What is a rectangle? What is a rational number? What is addition?). Defining format requires the student to categorize terms and list their essential characteristics, placing the student in the position of instructor to an audience seeking information.

Chapter 6: **Every Word Has a Story—Morphology and Etymology** helps students expand their mathematical vocabulary through the study of various forms of the terms they use and linguistic origins of these terms. Students learn that the language of mathematics is both specific to mathematics and part of our everyday life.

Chapter 7: **Frame Your Writing—Profiles and Frames** provides students with templates for gathering information and structural formats for organizing writing and enhancing communicative abilities. Profiles and Frames are two types of detailed outlines that give the student the organizational schema needed for different mathematical genres (e.g., problem solving, reasoning and proof, representation).

Chapter 8: **Think in Threes—Reasons, Procedures, Results** shows students how to compose problem-solving statements, give reasons and proofs, use persuasion related to mathematics, state results, and explain methodology. This extensive strategy provides students with the essential essay formats commonly used in mathematical writing and on standardized tests.

Chapter 9: **Meet a Mathematician—Who's Who in Mathematics** introduces students to accomplished mathematicians and their contributions to our mathematical knowledge. Students are exposed to mathematicians from different backgrounds and cultures and gain an understanding of the history of mathematics through human endeavor and knowledge. The strategy also shows students, through biographies, different careers related to mathematics.

Chapter 10: **Mathematics Takes You Places—Where in the World** relates the world of geography to the world of mathematics. From measuring distances in our neighborhood to understanding light years, there is an integral relationship between our world and the world of mathematics. Where in the World combines strategies to make students aware that mathematics is worldwide and universal and that mathematics can be used to understand where we have been, where we are now, and where we might be going.

Chapter 11: **Know Thyself—Personifications and Interactions** asks students to personify mathematical terms and concepts so that they can write about these terms and concepts in vivid and in-depth ways, focusing on significant details and accuracy of information. Through "interactions" with mathematical "personae,"

students become knowledgeable about the mathematical terms they are personifying and see mathematics from different perspectives, including humor.

Chapter 12: **Mathematically Literate—The Way to Go** summarizes how the 10 strategies can be used together to teach the mathematics and the communication skills students need to meet the standards for mathematics. The Planning Wheel has been completed.

Resources: At the request of many teachers, we have added rubrics and templates related to the strategies. Use them as appropriate for your own students and feel comfortable to create your own.

By following these strategies, you will help your students become good mathematical writers and better learners of mathematics. Before you begin instruction, use the Prologue (Figure 1.4) to explain to students what they will learn by becoming mathematicians of accomplishment.

Prologue for Students

Be a mathematician so that you will know

- words of mathematics,
- numerous symbols and signs,
- various types of numbers,
- careers and subjects that use mathematics,
- problems that mathematics can solve in your own life and in the world,
- tools used by mathematicians,
- mathematicians who have contributed to our knowledge,
- creative ideas that come from mathematicians and mathematics,
- games and lively activities that use mathematics.

Figure 1.4