2. AN OVERVIEW OF FUZZY SET MATHEMATICS

In this chapter, we provide a nontechnical introduction to fuzzy set mathematics. Rather than focusing on mathematical details, we will concentrate on making the concepts as clear as possible. There are several useful technical introductions in engineering textbooks, the most comprehensive being Zimmerman (1993) and Klir and Yuan (1995). Readers interested in more information about the topics covered in this chapter should consult these texts. Fuzzy set theory is a generalization of set theory. Although set theory is the foundation of the modern approach to mathematics and would be familiar to anyone with knowledge of, say, game theory or probability theory, we cannot assume that everyone is familiar with it. Thus, we will first start with a very brief overview of set theory and operations on sets. Then we will proceed to consider fuzzy sets as a particular extension of standard "crisp" set theory. Although "fuzzy" often carries a pejorative connotation, the mathematics of fuzzy set theory is precise. Its purpose is to allow us to better model phenomena that exhibit a certain kind of uncertainty, degree-vagueness.

2.1 Set Theory

The books mentioned above have reasonable introductions to set theory. Any introductory text on probability theory, real analysis, graph theory, logic, mathematical statistics, or linear algebra should also contain an introduction. Classical set theory is a mathematical calculus for dealing with collections of objects and certain relationships among these objects. At its most basic, a *set* is simply a list of objects, such as $A = \{a, b, c, d, e\}$ or $B = \{\text{orange, lemon, lime, grapefruit, tangerine}\}$. But sets generally become interesting by being connected with a *rule* that determines membership or nonmembership in the set. For instance, Set A can also be specified as the rule "first five letters of the alphabet" and Set B can be specified by the rule "commonly available citrus fruits," provided, of course, that "common" can be given precise meaning. Clearly, in any use of sets for modeling empirical reality or for testing real data, the rule connecting objects to each other is of utmost importance and must be specified clearly. In a location where kumquats and mandarins are grown in abundance but limes and grapefruit are unknown, the rule "common citrus fruits" would not specify Set B.

There are four common operations on sets: union, intersection, negation, and inclusion, commonly denoted by the symbols $\cup$, $\cap$, ~, and $\subset$, respectively (different notation for these operators is sometimes used by other authors). With these operations, it is possible to piece together quite complicated sets.

Union and intersection are known as connectives because they create a new set from two (or more) other sets according to a specified procedure. *Union* glues two sets together and corresponds to "or" in the inclusive sense, often expressed in natural language as and/or. Using the sets above, $A \cup B =$ {a, b, c, d, e, orange, lemon, lime, grapefruit, tangerine}. *Intersection* is the overlap between two sets and corresponds to "and." The two sets above have no elements in common and so their intersection is empty. We write $A \cap B = \varnothing$, the symbol for the null or empty set, the set that has no elements at all.

Negation, corresponding to "not," creates the complement of the set, which contains all elements in the universal set that are not in the set. Its definition requires that we define our universe of discourse, represented by the "universal" set *U.* Without *U,* we cannot meaningfully find a complement, and it is quite unclear whether we can make substantively meaningful statements about the sets at all. Assume for the moment that for Set *A* defined above, $U =$ {all letters of the English alphabet}. Then $\sim A =$ {f, g, . . . , z}. Note that $A \cup \sim A = U$, which says in words, "everything that is *A,* and everything that is not *A,* is everything." Also note that $A \cap \sim A = \varnothing$: "Nothing is in both *A* and not *A* at the same time." This statement is known as the Law of the Excluded Middle; it plays an important role in understanding fuzzy set theory because fuzzy intersections do not generally obey the Law.

Inclusion concerns whether a set has elements in common with another set. Set *P* is included in another set, *Q,* if all elements in *P* also are in *Q.* In the case of *A* and *B,* it is clear that neither set includes the other. However, given Set $T =$ {a, b, . . . j}, $A \subset T$ is read as "*A* is contained in *T*" or "*T* includes *A.*" As we shall see in Chapter 5, the asymmetry of inclusion is exceptionally useful for examining relationships between empirical cases that are quite different from the correlations typically employed by social scientists. Inclusion and intersection have a special relationship. When $P \subset Q$, then $P \cap Q = P$. When $P \subset Q$ and $Q \subset P$, then $P = Q$. See Table 2.1 for more on key set-theoretic operations.

2.2 Why Fuzzy Sets?

We define Set $V =$ {a, e, i, o, u}, the set of vowels. Logically, $C = \sim V$, the set of consonants, because a letter is either a consonant or a vowel.

TABLE 2.1
Key Set-Theoretic Operations

Operation	*Symbol*	*Notation*	*Verbal Translation*
Union	$\cup$	$A \cup B$	All elements in either *A* or *B*, or both
Intersection	$\cap$	$A \cap B$	Elements that are in both *A* and *B* only
Complement	~	~*A*	Elements in *U* that are not in *A*
Inclusion	$\subset$	$A \subset B$	All elements in *A* are also in *B*

However, we know that in English, the letter *y* is sometimes a vowel and sometimes a consonant. For example, in the word "my," *y* is a vowel, but in the word "yours," it is not. Does *y* belong in Set *V,* or does it belong in *C,* the set of consonants? The answer is unclear because *y* does not fit neatly into either *V* or *C,* but rather into both. This means, of course, that the rule separating vowels and consonants does not lead to a mutually exclusive classification of letters suggested by the dichotomy between vowels and consonants. The letter *y* violates the Law of the Excluded Middle that is assumed when we define $C = {\sim}V$.

It is difficult to think precisely about even this example familiar to children, but the problem resembles those faced every day in the process of constructing data sets and making inferences about objects in the data set. Classical set theory is often not adequate for dealing with uncertainty in the rule that assigns objects to sets. Mathematical objects generally can be defined precisely; empirical objects often cannot be so defined.

Fuzzy sets are designed to handle a particular kind of uncertainty—namely *degree-vagueness*—which results when we have a property that can be possessed by objects to varying degrees. Vagueness is easiest to see by referring to a classical paradox, the Sorites, an example of which we describe now. Consider a truckload of sand. Clearly, this constitutes a heap of sand. If we remove one grain of sand, the resulting pile is still a heap. Arguing by a possibly fallacious appeal to mathematical induction, we can remove another grain of sand and still have a heap. And so forth. Eventually, however, we have so little sand that no one would be willing to call whatever is left a heap. Thus, the definition of heap is not precise. It is subject to vagueness because nowhere in the process is there a point that divides things into two distinguishable states: heap and not-heap.

Many concepts in the social sciences contain essential vagueness in the sense that while we can define prototypical cases that fit the definition, it is not possible to provide crisp boundaries between sets. Consider poverty. Given a context, such as "single and lives in a college town in Midwestern U.S.A." (which provides an understanding of cost of living), we can define a poverty line relatively simply: "made less than $20,000 per year in 2003." Classical set theory would lead us to declare that a Midwesterner who made *exactly* $20,000 per year is, therefore, *not* poor, even though everyone would recognize that adding one extra dollar of income makes no material difference in the life of the person in question. However, adding $10,000 a year to the person's income would probably make her not poor. Thus, somewhere between an annual income of $20,000 and $30,000, the person would cease to be poor. Where exactly? Given a proposed boundary, we can almost always play the same game, noting that one more dollar does not make one go from being poor to not poor. Fuzzy set theory provides a mathematical toolbox for analyzing situations like this with precision, not via a definite cutoff, but by defining a *degree* of membership between the qualitatively different states of definitely poor and definitely not poor.

2.3 The Membership Function

A fuzzy set is based on a classical set, but it adds one more element: a numerical degree of membership of an object in the set, ranging from 0 to 1. Formally, the *membership function* m_A is a function over some space of objects Ξ mapping to the unit interval [0, 1], and the mapping is denoted by

$$m_A(x) : \Xi \rightarrow [0, 1].$$

This generates fuzzy set A. Note that a domain may refer to a "universal" set, but it also can be defined in terms of some mathematical region such as the real line or an interval representing the range of a scale.

The membership function is an index of "sethood" that measures the degree to which an object x is a member of a particular set. Unlike probability theory, degrees of membership do not have to add up to 1 across all objects, so many or few objects in the set could have high membership. However, an object's membership in a set and the set's complement must still sum to 1. The main difference between classical set theory and fuzzy set theory is that the latter admits to partial set membership. A classical or crisp set, then, is a fuzzy set that restricts its membership values to {0, 1}, the endpoints of the unit interval. Fuzzy set theory models vague phenomena by assigning any object a weight given by the value of the membership function, measuring the extent to which the rule "this object is in Set A" is judged to be true.

We use two simple examples to illustrate a number of points. First, we will construct a set of "common citrus fruits," assigning membership values subjectively by picking numbers from {0, .25, .50, .75, 1} based on our own

TABLE 2.2
Common Citrus Fruits

Fruit	*Membership*
Navel orange	1.00
Lemon	1.00
Red grapefruit	0.75
Lime	0.75
Tangerine	0.50
Kumquat	0.00
Mandarin	0.25

"expertise." These assignments are displayed in Table 2.2. Assignment of membership is a difficult problem that requires a lot of thought, and a great deal more will be said about the task in Chapter 3. The procedure just adopted is not all that different from the coding of objects as conducted by many social scientists working outside the context of fuzzy sets, however.

A useful notation is to generalize the list of elements notation for standard sets to a list of ordered pairs: {(Navel Orange, 1), (Lemon, 1), (Red Grapefruit, .75), (Lime, .75), (Tangerine, .5), (Kumquat, 0), (Mandarin, .25)}. The list of ordered pairs notation is compact and useful for relatively small sets.

Our second example illustrates a rule that takes the domain into membership. Typically, this is done when the domain is defined in terms of a quantitative construct. For instance, using the poverty example given above, we might decide to use a *linear filter* over income. In the definition below, membership in the set of poor people is 0 if annual income exceeds \$30,000, increases linearly for incomes ranging from \$30,000 down to \$20,000, and equals 1 for any income below \$20,000.

$$Poor(x) = \begin{cases} 0, & x > 30{,}000 \\ \dfrac{30{,}000 - x}{30{,}000 - 20{,}000}, & 20{,}000 \le x \le 30{,}000. \\ 1, & 0 \le x < 20{,}000 \end{cases}$$

This seems simple. However, one dilemma we must face immediately when constructing a fuzzy set (or indeed any set) is the definition of the universal set *U*. The observant reader may note that the fruit set defined above is confounded in that it includes at least two properties, "common" and "citrus." (This was intentional.) What constitutes the universe of discourse? Is it citrus fruits specifically, fruits in general, things found at a grocery store, or something else? A membership value takes on different meanings for different universal sets. In the case of $U = \{\text{citrus fruits}\}$, the kumquat's zero membership value indicates it is quite uncommon, although it is certainly a citrus fruit. If, however, $U = \{\text{fruits in general}\}$, many other cases would have zero membership by virtue of not being citrus fruits. Apples are certainly common, but they are not citrus fruits at all and thus fail on that criterion entirely. Even this seemingly trivial set is, in fact, rather complicated. If a reader walks away with nothing else, it should be a reminder that when constituting a population, clarity is essential.

2.4 Operations of Fuzzy Set Theory

Like classical set theory, fuzzy set theory includes operations union, intersection, complement, and inclusion, but also includes operations that have no classical counterpart, such as the modifiers concentration and dilation, and the connective fuzzy aggregation. In this section, all formulas are written with the assumption that only two sets are considered, but it is possible to extend all to three or more sets fairly easily by mathematical induction. To illustrate the fuzzy operations, we elaborate the fruits example. We have constructed four fuzzy sets over the universe of discourse "fruits"; this is not an exhaustive list. *Common* represents a subjective assessment of the degree of availability of fruits in an American supermarket. *Citrus* is true if the fruit in question is classified botanically as a citrus. *Rose* is true if the fruit in question is classified botanically as a rose. Finally, *Sour* represents the subjective degree of sour taste. *Citrus* and *Rose* are crisp sets because all membership values are either 0 or 1 (see Table 2.3).

Membership in the *fuzzy union* is defined as the maximum degree of membership in the sets. Membership in the union $X \cup Y$ may be written

$$m_{X \cup Y} = \max(m_X, m_Y).$$

Thus, the membership of an orange in the set *Common* $\cup$*Citrus* would be max(1.00, 1.00) = 1.00, and its membership in *Rose* $\cup$*Sour* would be max(.00, .25) = .25. Membership in the *fuzzy intersection* is defined as the minimum degree of membership in the sets, that is,

$$m_{X \cap Y} = \min(m_X, m_Y).$$

Thus, an orange's membership in the set "common and sour fruit" would be min(1.00, .25) = .25. The *fuzzy complement* is defined as $m_{\sim X} = 1 - m_X$. Thus an orange's membership in the set ~*Sour* is 1 – .25 = .75.

TABLE 2.3
Fruits Example With Membership of Two Derived Sets

Fruit	*Common*	*Citrus*	*Rose*	*Sour*	*Citrus∪Rose*	*Common∩Sour*
Navel orange	1.00	1.00	0.00	0.25	1.00	0.25
Lemon	1.00	1.00	0.00	1.00	1.00	1.00
Red grapefruit	0.75	1.00	0.00	0.75	1.00	0.75
Lime	0.75	1.00	0.00	0.75	1.00	0.75
Tangerine	0.50	1.00	0.00	0.25	1.00	0.25
Kumquat	0.00	1.00	0.00	0.00	1.00	0.00
Mandarin	0.25	1.00	0.00	0.00	1.00	0.00
Delicious apple	1.00	0.00	1.00	0.00	1.00	0.00
Star fruit	0.00	0.00	0.00	0.25	0.00	0.00
Banana	1.00	0.00	0.00	0.00	0.00	0.00
Red raspberry	0.75	0.00	1.00	0.75	1.00	0.75
Bing cherry	0.25	0.00	1.00	0.25	1.00	0.25
Strawberry	0.75	0.00	1.00	0.00	1.00	0.00
Coconut	0.50	0.00	0.00	0.00	0.00	0.00
Pineapple	0.50	0.00	0.00	0.50	0.00	0.50
Green grape	1.00	0.00	0.00	0.50	0.00	0.50

Unless otherwise noted, we use the max and min operators throughout this book for fuzzy union and fuzzy intersection, respectively. However, it is important to note that these are not the only definitions of the union and intersection suited to fuzzy set theory. Smithson (1987, Chapter 1) discusses this issue extensively, although most of the other books cited also have useful discussions in their consideration of t-norms and co-norms. In some contexts, alternative definitions of the operators are better able to meet the needs of particular applications. For example, the *product operators* are $m_{X \cup Y} = m_X + m_Y - m_X m_Y$ and $m_{X \cap Y} = m_X m_Y$. These formulas are, in fact, the same as the rules for compound independent events in probability theory. Unlike the max-min operations, they are continuous; changes in membership in one set are always reflected in the membership of the union or intersection. By contrast, this is not true for the max-min operators. This more continuous change may better reflect the underlying conceptual space.

Despite discontinuity, max-min operations remain the "industry standard." They are very easy to calculate, which is a virtue in some cases. Perhaps most importantly, they are relatively resistant to perturbations in the input membership values—which are often due more to measurement error than real variation—and demand only ordinal measurement. The multiplicity of operators is both a strength and a weakness of fuzzy set theory. As a strength, many different operators provide options for modeling different concepts. As a weakness, there are many choices to make, and it is not always clear which alternative is best. Of course, these operators all reduce to the classical ones when membership is restricted to just 0 and 1.

It is possible to chain operators together, thereby constructing quite complicated sets. In fact, much of the power of fuzzy set theory comes from this, as it is possible to derive many interesting sets from chains of rules built up from simple operators. The orange's membership in the set ~*Common*∩*Sour* would be min(1 – 1.00, .25) = 0. Indeed, as the orange is nearly prototypical of a sweet, common fruit, it should make sense that the membership in the set is low.

Fuzzy inclusion is somewhat more complicated. We introduce the Classical Inclusion Ratio (CIR) here, deferring a more complete discussion until Chapter 5. For crisp sets, inclusion is an all-or-nothing matter. Either Set *A* is included in Set *B* or it is not, and all it takes is one element in *A* not in *B* for inclusion to fail. This is decidedly unfuzzy and furthermore does not make sense from a data analytic standpoint, where we would expect to see some errors from the general pattern simply due to chance. Because crisp sets are just fuzzy sets with no membership values on the interior of the unit interval, they also have membership functions. Therefore, inclusion can be translated into a statement about membership: For *B* to include *A,* objects in *A* must have membership no greater than objects in *B*. We can easily extend this to continuous membership. Thus, the CIR simply counts the number of such objects relative to the total number of objects. If there are *n* objects,

$$\mathrm{CIR}_{A \subset B} = \#(m_A \leq m_B)/n. \qquad [2.1]$$

Because this is a proportion, it is possible to use the standard statistics for proportions to form the basis of statistical tests about the CIR, which is one of its main selling points. Another useful benchmark for inclusion of *A* in *B* is to consider how similar $m_{A \cap B}$ is to m_A, which can easily be seen by plotting $m_{A \cap B}$ on m_A. If identical, then they should form a straight line with intercept 0 and slope 1. In the fruits example, $\mathrm{CIR}_{Sour \subset Common} = 15/16 = .9375$, indicating that *Sour* is fuzzily included in *Common.*

Recall that fuzzy sets do not obey the Law of the Excluded Middle. Consider the lime, which has .75 membership in *Sour* and therefore a membership in ~*Sour* of .25. The membership in *Sour*∩~*Sour* is min(.75, .25) = .25. Given that we are considering a situation of vagueness, this seems sensible. But genetic engineering aside, a plant cannot be both citrus and not citrus at the same time, and the membership of the lime in "citrus and not citrus" is min(1, 0) = 0, as it should be.

We mentioned three other operations on fuzzy sets that are important: concentration, dilation, and aggregation. None of these operators has

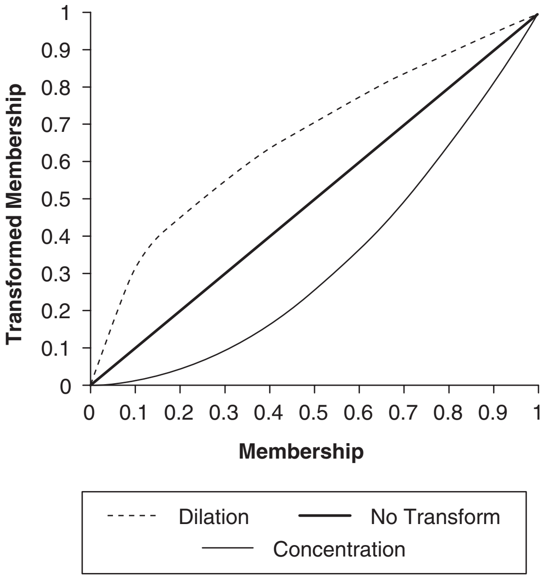

Figure 2.1 Concentration and Dilation

classical set parallels because all of them depend on membership values between 0 and 1. Concentration and dilation modify one set, similar to the complement, whereas aggregation is another connective between sets, similar to union and intersection. Concentration and dilation modify membership. Zadeh (1965) suggested that concentration corresponds to the phrase "very *X*," where *X* is the defining property, whereas dilation often is associated with the phrase "sort of *X*." The original *concentration operator* was $m_{Cx} = m_x^{\,2}$, and the original *dilation operator* was $m_{Dx} = m_x^{\,1/2}$. Generalization to using powers greater than one for concentration and powers less than one but greater than zero for dilation are straightforward.

The sense of these operators comes from the properties of power transformations of the unit interval: Power transformations in the unit interval map into the unit interval, which means that they can be interpreted as membership values. Concentration reduces all the values except 0 and 1 by squaring them, but the effect is weakest on those that are already small. Conversely, dilation increases all membership values except 0 and 1, but the effect is weakest on those that are already large. Figure 2.1 illustrates this,

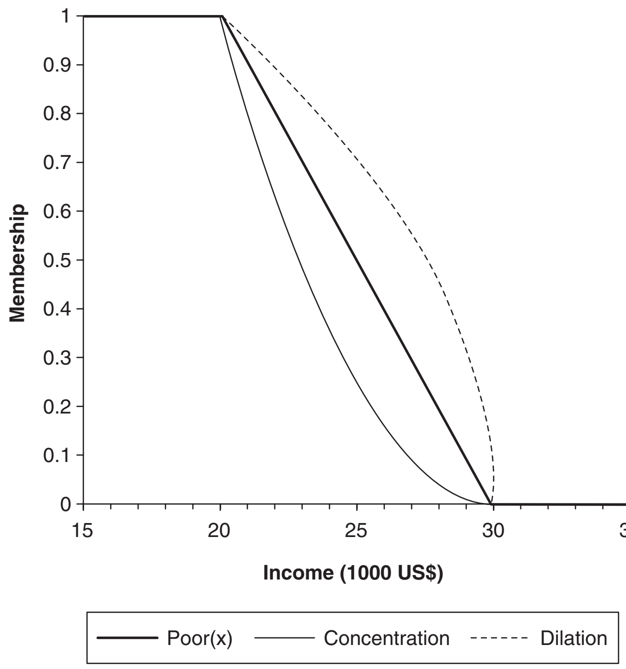

Figure 2.2 Concentration and Dilation for Membership Function *Poor*(*x*)

and Figure 2.2 shows concentration and dilation applied to the membership function *Poor*(*x*) given above.

The suitability of fuzzy set theory to model natural language usages of these terms, which are called linguistic hedges, has been questioned most forcefully by Lakoff (1973), particularly the use of dilation to model the hedge "sort of." Smithson (1987, Chapters 1–2) offers an extensive discussion drawing on the literatures in philosophy and cognitive science. However, we do not propose fuzzy set theory as a good model for natural language, but as a formal language for scientists operating in a domain of systematized logical reconstructions. The test of fuzzy set theory is whether it provides useful results. We discuss some further issues regarding transformations of fuzzy set memberships in Chapter 3 (Section 3.5 on sensitivity analysis) and present an example application of concentration and dilation in Chapter 6. One argument against their broad use is that they demand a high level of measurement, higher than most users will wish to assume.

The final fuzzy set operator we will discuss here is *fuzzy aggregation,* denoted by the symbol Γ (Gamma). Thus, the aggregation of two sets *X* and *Y* is denoted *X* Γ*Y,* and the aggregation of sets *X, Y,* and *Z* would be *X* Γ*Y* Γ*Z.* Classical sets have two connectives, unions and intersections, and these have extensions to fuzzy sets. As already discussed, membership in a union is determined by the maximum of memberships in the input sets, but membership in an intersection is determined by the minimum of memberships in the input sets. These are often described in terms of the strongest link/weakest link metaphor, because membership in the union is determined by the strongest link in the chain whereas membership in the intersection is determined by the weakest link in the chain. In this sense, fuzzy union is fully compensatory in that low membership values in Sets *A, B,* and *C* are completely compensated by a high membership in Set *D.* Fuzzy intersection is not at all compensatory, in that high membership values in Sets *A, B,* and *C* cannot compensate at all for a low membership in Set *D.* Alternatively, fuzzy union models redundant causation, whereas fuzzy intersection models conjoint causation.

However, in many cases, theory says that several properties contribute to overall membership in the aggregate, but that low values in one property are not fully compensated for by high values in another, invalidating fuzzy union. In fact, this is very similar to the assumption often used in scale construction, where different components make up the whole by summing together. There are many different aggregation operators, but we will discuss two simple ones. The first is the geometric mean of membership functions

$$m_{x\Gamma y} = \sqrt{m_x m_y}.$$

The geometric mean acts like an average for membership values near each other but like the intersection when one of the membership values is close to zero. The second is the arithmetic mean of the union and intersection

$$m_{x\Gamma y} = \frac{\max(m_x, m_y) + \min(m_x, m_y)}{2}.$$

In the two-set case, this is just the arithmetic mean of the membership values, but when three or more sets are considered, that is not necessarily true. More sophisticated aggregation operators—of which the ones listed above are special cases—are discussed in great detail in Zimmerman (1993). If we wanted to aggregate *Common* and *Sour* via the geometric mean, for the orange we would get

$$\sqrt{1.00 \times 0.25} = .50,$$

whereas the arithmetic mean would be $(1.00 + .25)/2 = .675$. Interpretation of aggregations is, of course, a matter for substantive theory.

2.4.1 Level Sets

Level sets provide a useful connection between crisp sets and fuzzy sets. Starting with a fuzzy set X, we introduce a level parameter, $\lambda \in [0,1]$, and define a set as $Y_\lambda = \{x \in X \mid m_x \geq \lambda\}$. Translating into words, Y_λ is the classical (dichotomous) set made from fuzzy set X with elements that have membership greater than λ. For instance, if $X = \{(a,0), (b,.2), (c,.3), (d,.6), (e,.8), (f,1)\}$, then $Y_0 = \{a, b, c, d, e, f\}$, $Y_{.5} = \{d, e, f\}$, and $Y_1 = \{f\}$. Notice that if $\lambda > \theta$, then $Y_\lambda \subseteq Y_\theta$, as can be seen from the example as $Y_1 \subset Y_{.5} \subset Y_0$. One use of level sets is to generate contingency tables. Using the fruit example, cross-tabulating $Common_{.5}$ and $Sour_{.5}$ produces Table 2.4. We will use level sets extensively in Chapter 5.

TABLE 2.4
Cross-Classification Generated by Level Sets

	$Common_{.5} = 0$	$Common_{.5} = 1$
$Sour_{.5} = 1$	0	6
$Sour_{.5} = 0$	4	6

2.5 Fuzzy Numbers and Fuzzy Variables

Is "several" a number? It clearly specifies numerical information, but it is vague. Instead, it specifies a range of possible integers, some of which are more plausible than others as referring to "several." "Several" does not really apply to just one integer, instead it is vague. Verbal quantifiers such as "several" can be made precise using fuzzy set theory by creating a *fuzzy number.* Smithson (1987) provided the following data taken from a survey of 23 undergraduates used to specify meaning for the term "several," shown in Figure 2.3, which plots the mean membership value and ±2SE (truncated to lie in the unit interval). Subjects were asked to give a numerical membership rating for each number on a bounded response scale that was subsequently mapped into the unit interval. Clearly, the peak is over 6, and subjects judged that the numbers 5 to 8 are the most consistent with "several." Similarly, Budescu and his colleagues (e.g., Budescu, Karelitz, & Wallsten, 2003) have used fuzzy numbers to elicit probability assessments. This process is important in risk analysis and other areas where data from

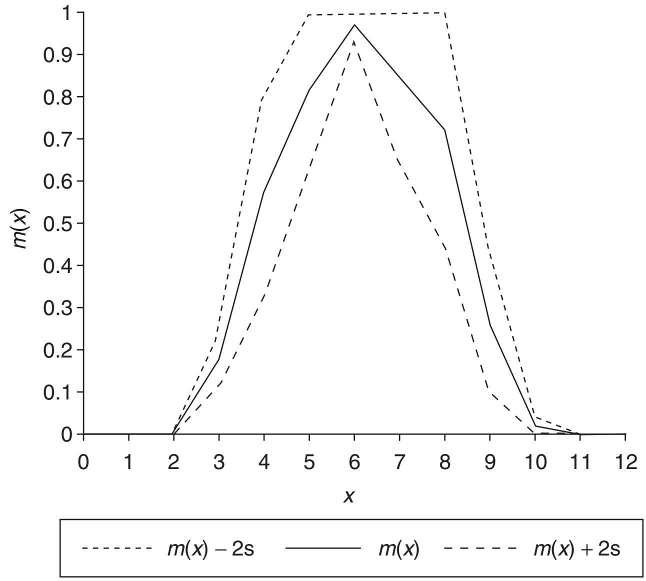

Figure 2.3 Fuzzy Number "Several"

judges are frequently stated in natural language, but it is desirable to have a numerical answer with an understood degree of uncertainty. By making more precise the typical understanding of a term like "highly unlikely," fuzzy variables provide a means to translate qualitative language into quantitative statements.

The idea can be generalized to the notion of a *fuzzy variable.* For example, a survey question about sexual activity in teens might ask, "How many days in the past month did you have sex?" The response options might be {none, a few times, several times, many times}. The usual approach would be to break these categories into disjoint intervals, as in {0, [1,4], [5,8], [9,30]}. Many others are possible, and the results of the analysis could depend in nontrivial ways on the assignment of numerical values to the qualitative responses. Instead, we could use an approach similar to the one above for creating the fuzzy number "several" for the other responses, which would give us a better idea of exactly how many days the subjects had sex. Figure 2.4 shows a fuzzy variable constructed for this response set by mapping to the integers 0 to 30. Note that the intervals overlap, indicating degrees of uncertainty about exactly what the terms mean.

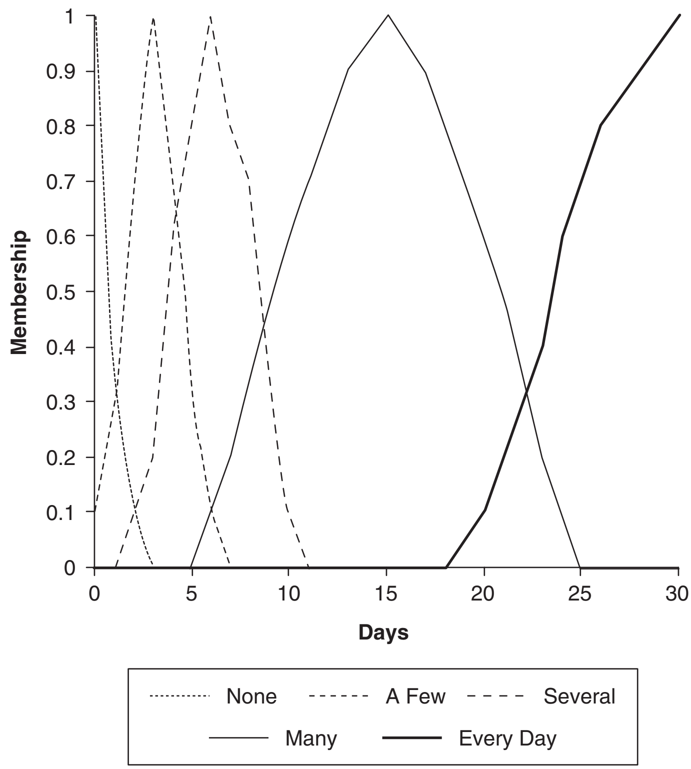

Figure 2.4 Fuzzy Variable "Number of Days in the Past Month"

2.6 Graphical Representations of Fuzzy Sets

Visualization is a key component of any data analysis, and data analysis with fuzzy sets is no exception. The first step in any analysis should be to graph the data. We will discuss graphs that consider only one fuzzy set at a time, focusing on it and its domain, and then discuss bivariate graphs, where we examine the membership of a domain of objects that are members of two fuzzy sets. The usual caveats and guidelines for creating good graphics apply. We refer readers to Jacoby (1997, 1998) or Cleveland (1993) for useful discussions. Because a membership function is a numerical value in the unit interval, we can graph it over its domain. Obviously, if the domain set has more structure—for example, it is numerical—the plot will have more structure, as in the plot for "several" (Figure 2.3) in the